Tensor categories

Jonathan Gruber

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Introduction

In many fields of mathematics, one is naturally led to study tensor products of certain objects (e.g. sheaves in algebraic geometry, cobordisms in topology, modules in commutative algebra). All of these notions fit into the framework of *monoidal categories*, which gives an abstract definition of what a tensor product structure on a category should be. The aim of this course is to give an introduction to monoidal categories and tensor categories, the latter being certain monoidal categories endowed with some extra structure. (They are abelian and have a linear structure that is compatible with the tensor product, over some field.)

The first half of the course deals mainly with category-theoretical notions, starting from the definition of a monoidal category and then discussing important additional properties such as rigidity (the existence of duals) and braidings (functorial isomorphisms $X \otimes Y \cong Y \otimes X$). In the second half, we will turn our attention to tensor categories. Our prime example is the category of $\mathbf{Rep}_{\Bbbk}(G)$ finite-dimensional representations of a group G over a field $\Bbbk$, and we will discuss reconstruction theorems that allow us to recover a group (or Hopf algebra) from the corresponding category of representations, together with its monoidal structure and the *forgetful functor* that sends a representation to the underlying vector space. This gives rise to bijections between certain types of groups and Hopf algebras, up to isomorphism, and certain kinds of tensor categories, up to monoidal equivalence, which are broadly referred to as *Tannaka duality.*

Author's note

These notes are the my own synopsis of material that has been collected from many different sources, but most importantly, from [EGNO15]. Further important references include [ML98, DM82, EGNO09] and some websites such as nLab, MathStackExchenge, and Wikipedia. No originality is claimed, except in the presentation of the material, and all mistakes should be considered my responsibility.

These notes are also a work in progress. If you find any mistakes or typos and if you have comments or suggestions, please let me know.

I would like to thank Johannes Flake for helpful discussions and encouragement, and for making his own notes available.

1 Monoidal categories

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) a class $\mathrm{Ob}(\mathcal{C})$ of *objects* of $\mathcal{C}$;

(2) for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of *homomorphisms* from A to B;

(3) for every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition map*

$$\circ \colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

such that the following axioms hold:

(a) for $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f \colon X \to Y$, $g \colon Y \to Z$, $h \colon Z \to W$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

(b) for $X \in \mathrm{Ob}(\mathcal{C})$, there exists an *identity homomorphism* id_X such that $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and homomorphisms $f \colon Y \to X$ and $g \colon X \to Y$.

Remark 1.2. (1) The homomorphisms in a category are often simply referred to as *morphisms.*

(2) As in points (a) and (b) of the definition, we often denote a homomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ by an arrow $f \colon X \to Y$.

(3) A morphism $f \colon X \to Y$ is called an *isomorphism* if there exists a morphism $g \colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In that case, we write $g = f^{-1}$ and $X \cong Y$.

Example 1.3. We list some important examples of categories:

- **Set**: the category of sets, with maps between sets as homomorphisms;
- **Grp**: the category of groups with group homomorphisms;
- **AbGrp**: the category of abelian groups with group homomorphisms;
- $\mathbf{Vect}_{\Bbbk}$: the category of finite-dimensional $\Bbbk$-vector spaces with $\Bbbk$-linear maps, for a given field $\Bbbk$;
- $A-\mathbf{Mod}$: the category of A-modules with A-module homomorphisms, for a given algebra A; we write $A-\mathbf{mod}$ for the subcategory of finite-dimensional A-modules.

Definition 1.4. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns

(1) to every object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$;

(2) to every homomorphism $f\colon X \to Y$ in $\mathcal{C}$ a homomorphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$;

in such a way that
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \qquad \text{and} \qquad F(f \circ g) = F(f) \circ F(g).$$

Remark 1.5. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the product category $\mathcal{C} \times \mathcal{D}$ whose objects are the pairs (X, Y) of objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$, and where

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\big((X,Y),(Z,W)\big) = \mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{D}}(Y,W)$$

for $X, Z \in \mathrm{Ob}(\mathcal{C})$ and $Y, W \in \mathrm{Ob}(\mathcal{D})$. The composition and the identity morphisms are defined component-wise in the obvious way. A functor from $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ to some other category $\mathcal{E}$ is often called a *bifunctor*.

Definition 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta\colon F \to G$ is a family of morphisms $\eta_A\colon F(A) \to G(A)$ in $\mathcal{D}$, for every object A of $\mathcal{C}$, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\ \eta_A\ } & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\ \eta_B\ ]{} & G(B) \end{array}$$

For an object A of $\mathcal{C}$, we call η_A the component of η at A. A natural transformation is called a *natural isomorphism* if all of its components are isomorphisms.

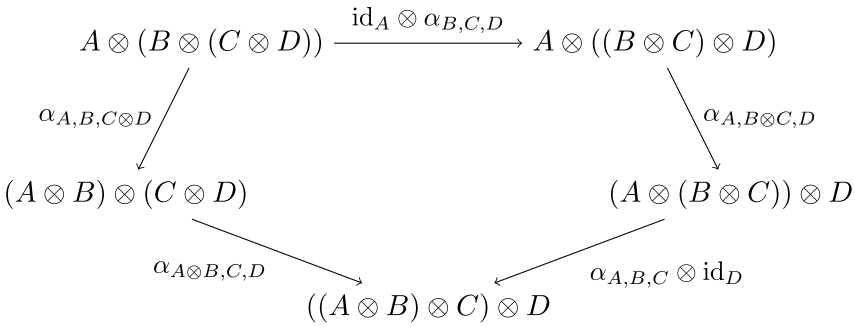
Remark 1.7. The functors between two categories $\mathcal{C}$ and $\mathcal{D}$ form a category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ whose morphisms are natural transformations between functors. The composition of natural transformations is defined componentwise.

Definition 1.8. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an equivalence if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Definition 1.9. A *monoidal category* is a tuple $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, where

- $\mathcal{C}$ is a category,
- $\mathbf{1}$ is an object of $\mathcal{C}$, called the *unit object*,
- $\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor called the *tensor product*,
- $a\colon -\otimes(-\otimes-) \to (-\otimes-)\otimes-$ is a natural isomorphism, called the *associativity constraint*,
- $\lambda\colon \mathbf{1}\otimes - \to \mathrm{id}_{\mathcal{C}}$ and $\rho\colon -\otimes\mathbf{1} \to \mathrm{id}_{\mathcal{C}}$ are natural isomorphisms, called the (left and right) *unitors*,

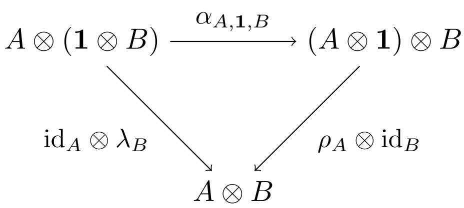
subject to the following axioms:
Pentagon axiom: For all objects A, B, C, D of $\mathcal{C}$, the following diagram commutes:



In other words, we have

$$\big(\alpha_{A,B,C\otimes D}\big) \circ \big(\alpha_{A,B\otimes C,D}\big) \circ \big(\mathrm{id}_A \otimes \alpha_{B,C,D}\big) = \big(\alpha_{A\otimes B,C,D}\big) \circ \big(\alpha_{A,B,C} \otimes \mathrm{id}_D\big).$$

Unit axiom / triangle axiom: For all objects A, B of $\mathcal{C}$, the following diagram commutes:



In other words, we have

$$(\rho_A \otimes \mathrm{id}_B) \circ \alpha_{A,\mathbf{1},B} = \mathrm{id}_A \otimes \lambda_B.$$

Remark 1.10. (1) When no confusion is possible, we simply write $\mathcal{C}$ instead of $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$. We also say that $(\mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is a *monoidal structure* on the category $\mathcal{C}$.

(2) Being monoidal is **not** a property of a given category, but an *additional structure*. A category can admit more than one monoidal structure. (See the examples below.)

(3) The fact that $\otimes$ is a bifunctor means that for suitable morphisms a, b, c, d in $\mathcal{C}$, we have

$$(a \otimes c) \circ (b \otimes d) = (a \circ b) \otimes (c \circ d).$$

(4) Instead of assuming the existence of $\mathbf{1}$ with natural transformations λ and ρ subject to the unit axiom, one can start from the (seemingly weaker, but actually equivalent) assumptions that there exists an object $\mathbf{1}$ with an isomorphism $\iota\colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ such that the functors $\mathbf{1} \otimes -$ and $- \otimes \mathbf{1}$ are equivalences. (No additional assumptions on the isomorphism ι are necessary.) See Sections 2.1 and 2.2 of [EGNO15] for more details.

Example 1.11. (1) The category **Set** of sets has a monoidal structure where the tensor product is given by the cartesian product and the unit object is a singleton $\{\bullet\}$.

(2) The category **Grp** of groups has a monoidal structure where the tensor product is given by the cartesian product and the unit object is the trivial group $\{1\}$.

(3) The category **AbGrp** of abelian groups has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\mathbb{Z}} -$ and the unit object is the group $\mathbb{Z}$ of integers. It also inherits a different monoidal structure from the category **Grp**; see the previous point.

(4) The category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$ has a monoidal structure where the tensor product is given by the usual tensor product $- \otimes_{\Bbbk} -$ and the unit object is the one-dimensional vector space $\Bbbk$.

(5) The category $\mathbf{Rep}_{\Bbbk}(G)$ of finite dimensional representations of a group G over a field $\Bbbk$ admits a monoidal structure where the tensor product is the usual tensor product of representations and the unit object is the trivial one-dimensional representation $\Bbbk$. More precisely, if we identify $\mathbf{Rep}_{\Bbbk}(G)$ with the category of finite-dimensional modules over the group algebra $\Bbbk[G]$ then the action of $\Bbbk[G]$ on the tensor product $M \otimes N$ of two $\Bbbk[G]$-modules M and N is uniquely determined by $g \cdot (m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$.

In the following, we refer to the objects of $\mathbf{Rep}(G)$ as *G-modules*.

(6) For a category $\mathcal{C}$, the category $\mathbf{End}(\mathcal{C}) = \mathbf{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors of $\mathcal{C}$ has a monoidal structure, where the tensor product is given by the composition of functors and the unit object is the identity functor $\mathrm{id}_{\mathcal{C}}$. The associativity constraints and unitors are identity natural transformations.

The two next examples will seem quite trivial for now, but they will become more interesting later when we add extra structure:

(7) Let G be a monoid and A an abelian group. Then we can define a monoidal category $\mathcal{C}_A^G$ with objects $\mathrm{Ob}(\mathcal{C}_A^G) = \{\delta_g \mid g \in G\}$ indexed by G and homomorphisms

$$\mathrm{Hom}_{\mathcal{C}_A^G}(\delta_g, \delta_h) = \begin{cases} A & \text{if } g = h, \\ \varnothing & \text{otherwise,} \end{cases}$$

for $g, h \in G$. The tensor product is defined by $\delta_g \otimes \delta_h = \delta_{gh}$ and by $a \otimes a' = aa' \in A$ for $g, h \in G$ and $a, a' \in A$, the unit object is δ_e (where $e \in G$ is the unit element) and the associativity constraints and unitors are identity maps.

(8) Let G be a monoid and let $\mathbf{Vect}_{\Bbbk}^G$ be the category of finite-dimensional G-graded $\Bbbk$-vector spaces $V = \bigoplus_{g \in G} V_g$, with homomorphisms given by grading-preserving linear maps. (That is, for $V = \bigoplus_g V_g$ and $W = \bigoplus_g W_g$ two G-graded vector spaces, the homomorphisms from V to W in $\mathbf{Vect}_{\Bbbk}^G$ are the linear maps $f \colon V \to W$ that satisfy $f(V_g) \subseteq W_g$ for all $g \in G$.) Then the monoidal structure on $\mathbf{Vect}_{\Bbbk}$ induces a monoidal structure $\mathbf{Vect}_{\Bbbk}^G$, where the grading on the tensor product of G-graded vector spaces V and W is given by

$$(V \otimes W)_g = \bigoplus_{hh'=g} V_h \otimes W_{h'}$$

for $g \in G$. The tensor product of two homomorphisms in $\mathbf{Vect}_{\Bbbk}^G$ is just the usual tensor product of linear maps. The unit object is the one-dimensional vector space $\Bbbk = \Bbbk_e$ whose unique non-zero grading piece is indexed by the unit object $e \in G$. The associativity constraint and the unitors come from the category $\mathbf{Vect}_{\Bbbk}$.

Observe that there is a faithful functor $i_{\Bbbk}^G \colon \mathcal{C}_{\Bbbk^\times}^G \to \mathbf{Vect}_{\Bbbk}^G$ with $i_{\Bbbk}^G(\delta_g) = \Bbbk_g$ the one-dimensional vector space with grading concentrated in degree g, for $g \in G$, and with the obvious definition on homomorphisms. This functor is compatible with the tensor product (up to a natural isomorphism); it is an example of a *monoidal functor* (to be defined shortly).

In our final example, we demonstrate that there are monoidal categories with a less obvious choice of associativity constraint.

(9) Let G be a monoid, let A an abelian group and let ω be a 3-cocycle for G with values in A, i.e. a map $\omega \colon G^{\times 3} \to A$ with

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_2, g_3, g_4)\omega(g_1, g_2g_3, g_4)\omega(g_1, g_2, g_3) \tag{1.1}$$

for $g_1, g_2, g_3, g_4 \in G$. Then we can define a monoidal category $\mathcal{C}_A^{G,\omega}$ with underlying category $\mathcal{C}_A^G$ and with tensor product and unit object defined as in point (7), but with associativity constraint α^ω defined by

$$\alpha^\omega_{g,h,k} = \omega(g,h,k)\colon \delta_g \otimes (\delta_h \otimes \delta_k) = \delta_{ghk} \longrightarrow \delta_{ghk} = (\delta_g \otimes \delta_h) \otimes \delta_k$$

for $g, h, k \in G$. Observe that the 3-cocycle condition implies that $\mathcal{C}_A^{G,\omega}$ satisfies the pentagon axiom. (In fact, a map $\omega\colon G^{\times 3} \to A$ defines an associativity constraint for $\mathcal{C}_A^G$ if and only if ω is a 3-cocycle.) The unitors are defined by $\lambda_g = \omega(e,e,g)$ and $\rho_g = \omega(g,e,e)^{-1}$ for $g \in G$, and the unit axiom becomes the equation $\omega(g,e,h) = \omega(g,e,e) \cdot \omega(e,e,h)$ for $g, h \in G$ (which also follows from (1.1) by setting $g_2 = g_3 = e$).

Given a 3-cocycle $\omega\colon G^{\times 3} \to \mathbb{k}^\times$ with values in the multiplicative group of a field $\mathbb{k}$, we can extend the associativity constraint α^ω on $\mathcal{C}^G_{\mathbb{k}^\times}$ to an associativity constraint α^ω on $\mathbf{Vect}_{\mathbb{k}}^G$ via

$$\alpha^\omega_{\mathbb{k}_g, \mathbb{k}_h, \mathbb{k}_k} = \omega(g,h,k) \cdot \alpha_{\mathbb{k}_g, \mathbb{k}_h, \mathbb{k}_k},$$

for $g, h, k \in G$, extended by additivity, where α denotes the 'usual' associativity constraint in $\mathbf{Vect}_{\mathbb{k}}^G$. (Note that every object of $\mathbf{Vect}_{\mathbb{k}}^G$ is a direct sum of objects of the form $\mathbb{k}_g$ with $g \in G$.)

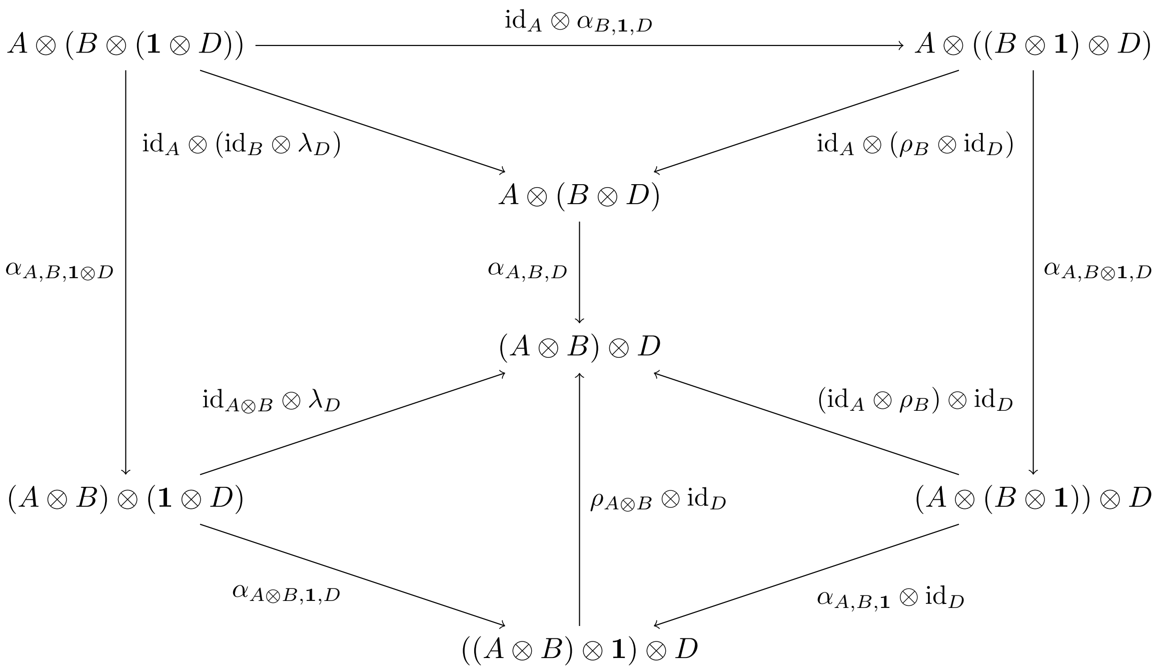
Remark 1.12. Given a monoidal category $\mathcal{C} = (\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$, we define the *opposite monoidal category* $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}, \mathbf{1}, \otimes^{\mathrm{op}}, \alpha^{\mathrm{op}}, \rho, \lambda)$ with the same underlying category, but tensor product defined by $X \otimes^{\mathrm{op}} Y = Y \otimes X$ and $f \otimes^{\mathrm{op}} g = g \otimes f$ for objects X, Y and homomorphisms f, g in $\mathcal{C}$, and with associativity constraint given by $\alpha^{\mathrm{op}}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$ for objects X, Y, Z of $\mathcal{C}$.

This is not to be confused with the *reverse category* $\mathcal{C}^{\mathrm{rev}}$ with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. The latter can also be endowed with a canonical monoidal structure. Note that $\mathcal{C}^{\mathrm{rev}}$ is often also called the opposite category of $\mathcal{C}$; we use non-standard terminology here to avoid confusion with the opposite monoidal category defined above.

Lemma 1.13. *For all objects A, B of $\mathcal{C}$, we have*

$$\rho_{A\otimes B} \circ \alpha_{A,B,\mathbf{1}} = \mathrm{id}_A \otimes \rho_B \qquad \textit{and} \qquad \lambda_{A\otimes B} \circ \alpha_{\mathbf{1},A,B} = \lambda_A \otimes \mathrm{id}_B.$$

Proof. Consider the following diagram, where all arrows are isomorphisms:



The external pentagon commutes by the pentagon axiom, the quadrangles commute by naturality of the associativity constraint α, and the top triangle and the bottom left triangle commute by the triangle axiom. Since all arrows are isomorphisms, this implies that the bottom right triangle commutes. Setting $D = \mathbf{1}$ and using the natural isomorphism $\rho\colon -\otimes\mathbf{1} \to \mathrm{id}_{\mathcal{C}}$, it follows that the following diagram commutes:

$$\begin{array}{ccc} A\otimes(B\otimes\mathbf{1}) & \xrightarrow{\alpha_{A,B,\mathbf{1}}} & (A\otimes B)\otimes\mathbf{1} \\ & {\scriptstyle \mathrm{id}_A\otimes\rho_B}\searrow \quad \swarrow{\scriptstyle \rho_{A\otimes B}} & \\ & A\otimes B & \end{array}$$

This proves the first claim, the second claim can be proven analogously. □

Definition 1.14. Let $(\mathcal{C},\mathbf{1},\otimes,\alpha,\lambda,\rho)$ and $(\mathcal{C}',\mathbf{1}',\otimes',\alpha',\lambda',\rho')$ be two monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a triple (F,φ,ε), where $F\colon\mathcal{C}\to\mathcal{C}'$ is a functor, $\varphi\colon F(-\otimes-)\to F(-)\otimes' F(-)$ is a natural isomorphism and $\varphi\colon F(\mathbf{1})\to\mathbf{1}'$ is an isomorphism, such that the following diagrams commute:

$$\begin{array}{ccc} F\big(A\otimes(B\otimes C)\big) & \xrightarrow{F(\alpha_{A,B,C})} & F\big((A\otimes B)\otimes C\big) \\ \downarrow{\scriptstyle \varphi_{A,B\otimes C}} & & \downarrow{\scriptstyle \varphi_{A\otimes B,C}} \\ F(A)\otimes' F(B\otimes C) & & F(A\otimes B)\otimes' F(C) \\ \downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes\varphi_{B,C}} & & \downarrow{\scriptstyle \varphi_{A,B}\otimes\mathrm{id}_{F(C)}} \\ F(A)\otimes'\big(F(B)\otimes' F(C)\big) & \xrightarrow[\alpha'_{F(A),F(B),F(C)}]{} & \big(F(A)\otimes' F(B)\big)\otimes' F(C) \end{array}$$

$$\begin{array}{ccc} F(A\otimes\mathbf{1}) & \xrightarrow{\varphi_{A,\mathbf{1}}} & F(A)\otimes' F(\mathbf{1}) \\ {\scriptstyle F(\rho_A)}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{F(A)}\otimes'\varepsilon} \\ F(A) & \xleftarrow[\rho'_{F(A)}]{} & F(A)\otimes'\mathbf{1}' \end{array} \qquad \begin{array}{ccc} F(\mathbf{1}\otimes A) & \xrightarrow{\varphi_{\mathbf{1},A}} & F(\mathbf{1})\otimes' F(A) \\ \downarrow{\scriptstyle F(\lambda_A)} & & \downarrow{\scriptstyle \varepsilon\otimes'\mathrm{id}_{F(A)}} \\ F(A) & \xleftarrow[\lambda'_{F(A)}]{} & \mathbf{1}'\otimes' F(A) \end{array}$$

A *monoidal natural transformation* between monoidal functors (F,φ,ε) and $(F',\varphi',\varepsilon')$ is a natural transformation $\psi\colon F\to F'$ such that

$$\varphi'\circ\psi_{-\otimes-} = \psi\otimes'\psi\circ\varphi \qquad \text{and} \qquad \varepsilon = \varepsilon'\circ\psi_{\mathbf{1}},$$

i.e. the following diagrams commute for all objects A,B of $\mathcal{C}$:

$$\begin{array}{ccc} F(A\otimes B) & \xrightarrow{\varphi_{A,B}} & F(A)\otimes' F(B) \\ {\scriptstyle \psi_{A\otimes B}}\downarrow & & \downarrow{\scriptstyle \psi_A\otimes'\psi_B} \\ F'(A\otimes B) & \xrightarrow[\varphi'_{A,B}]{} & F'(A)\otimes' F'(B) \end{array} \qquad \begin{array}{ccc} F(\mathbf{1}) & \xrightarrow{\psi_{\mathbf{1}}} & F'(\mathbf{1}) \\ & {\scriptstyle \varepsilon}\searrow & \downarrow{\scriptstyle \varepsilon'} \\ & & \mathbf{1}' \end{array}$$

Remark 1.15. (1) Being monoidal for a functor is an additional structure, and not a property. However, being monoidal for a natural transformation is a property.

(2) The composition of two monoidal functors $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ and $(F', \varphi', \varepsilon')\colon \mathcal{D} \to \mathcal{E}$ can be considered as a monoidal functor with structure maps defined as follows, for objects X, Y of $\mathcal{C}$:

$$F'\big(F(X \otimes Y)\big) \xrightarrow{F'(\varphi_{X,Y})} F'\big(F(X) \otimes F(Y)\big) \xrightarrow{\varphi'_{F(X),F(Y)}} F'\big(F(X)\big) \otimes F'\big(F(Y)\big)$$

$$F'\big(F(\mathbf{1})\big) \xrightarrow{F'(\varepsilon)} F'(\mathbf{1}) \xrightarrow{\varepsilon'} \mathbf{1}.$$

(3) Given a monoidal functor $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ such that F is an equivalence of categories, one can show that it is possible to choose a monoidal functor $(G, \psi, \epsilon)\colon \mathcal{D} \to \mathcal{C}$ such that there are monoidal natural isomorphisms $F \circ G \to \mathrm{id}_{\mathcal{D}}$ and $G \circ F \to \mathrm{id}_{\mathcal{C}}$. In that case, we say that $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent*. For more details, see Proposition 4.4.2 in [SR72].

Example 1.16. (1) For a group G and a field $\Bbbk$, the forgetful functor $F\colon \mathbf{Rep}_{\Bbbk}(G) \to \mathbf{Vect}_{\Bbbk}$ which sends a G-module to the underlying vector space is monoidal. (The structure maps φ and ε are identity maps.) For a G-module V and for $g \in G$, let us write $\varphi(g)_V \in \mathrm{End}_{\Bbbk}(V)$ for the action of g on V. Then $\varphi(g)$ defines a natural transformation from the functor F to itself; we write $\varphi(g) \in \mathrm{End}(F)$. (This follows from the fact that for a homomorphism $f\colon V \to W$ of G-modules, the equality $\varphi(g)_W \circ F(f) = F(f) \circ \varphi(g)_V$ is equivalent to $g \cdot f(v) = f(g \cdot v)$ for $v \in V$.) The natural transformation $\varphi(g)$ is monoidal because $\varphi(g)_{V \otimes W} = \varphi(g)_V \otimes \varphi(g)_W$ and $\varphi_{\Bbbk}(g) = \mathrm{id}_{\Bbbk}$, by definition of the tensor product of G-modules and of the trivial G-module. This example will play an important role later in the course.

Conversely, every vector space can be considered as a G-module with the trivial action of G, and this gives rise to a monoidal functor $e\colon \mathbf{Vect}_{\Bbbk} \to \mathbf{Rep}_{\Bbbk}(G)$.

(2) For a commutative ring R, there is a functor $F\colon \mathbf{Set}^{\mathrm{rev}} \to R-\mathbf{Mod}$ that sends a set X to the free R-module $F(X) \coloneqq R^X = \mathrm{Map}(X, R)$. At the level of homomorphisms, we define $F(f)$ via $g \mapsto g \circ f$, for maps $f\colon X \to Y$ and $g\colon X \to R$. This functor is monoidal with respect to the monoidal structure on **Set** via the Cartesian product and on $R-\mathbf{Mod}$ via the usual tensor product of R-modules.

(3) The total cohomology functor $H^*\colon \mathbf{coch}(\mathbf{Vect}_{\Bbbk}) \to \mathbf{Vect}_{\Bbbk}^{\mathbb{Z}}$ from the category of cochain complexes of $\Bbbk$-vector spaces to the category of graded $\Bbbk$-vector spaces is monoidal with respect to the usual derived tensor product on $\mathbf{coch}(\mathbf{Vect}_{\Bbbk})$ by the Künneth theorem: For two cochain complexes $X_\bullet$ and $Y_\bullet$, we have

$$H^i(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{j+k=i} H^j(X_\bullet) \otimes H^k(Y_\bullet),$$

matching the definition of the tensor product of $\mathbb{Z}$-graded vector spaces.

(4) Let G and H be groups, let A be an abelian group and let $\omega\colon G^{\times 3} \to A$ and $\pi\colon H^{\times 3} \to A$ be 3-cocycles. Suppose that there is a monoidal functor $(F, \varphi, \varepsilon)$ from $\mathcal{C}_A^{G,\omega}$ to $\mathcal{C}_A^{H,\pi}$, for some $\varphi\colon -\otimes- \to -\otimes-$ and $\varepsilon \in \mathrm{End}_{\mathcal{C}_A^G}(\delta_e) = A$. Then F defines a map $f\colon G \to H$ via $F(\delta_g) = \delta_{f(g)}$ for $g \in G$, and f is a homomorphism because

$$\delta_{f(gh)} = F(\delta_{gh}) = F(\delta_g \otimes \delta_h) \cong F(\delta_g) \otimes F(\delta_h) = \delta_{f(g)} \otimes \delta_{f(h)} = \delta_{f(g)f(h)}$$

for $g, h \in G$. Furthermore, φ defines a map $\varphi\colon G \times G \to A$ via

$$A \ni \varphi(g, h) \coloneqq \varphi_{g,h}\colon F(\delta_{gh}) = F(\delta_g \otimes \delta_h) \to F(\delta_g) \otimes F(\delta_h) = \delta_{f(g)}\delta_{f(h)} = \delta_{f(gh)}$$

for $g, h \in G$, and by the definition of monoidal functors, we have

$$\varphi(g,h)\varphi(gh,k)\omega(g,h,k) = \underbrace{\pi\big(f(g), f(h), f(k)\big)}_{=f^*\pi(g,h,k)}\varphi(h,k)\varphi(g,hk)$$

for all $g, h, k \in G$, that is

$$f^*\pi^{-1}\omega(g,h,k) = \varphi(h,k)\cdot\varphi(g,hk)\cdot\varphi(g,h)^{-1}\cdot\varphi(gh,k)^{-1} = d^2\varphi(g,h,k).$$

In other words, the 3-cocycle $\omega f^*\pi^{-1} = d^2\varphi$ is a 3-*coboundary*, so ω and $f^*\pi$ define the same element of the third cohomology group $H^3(G, A)$. (The latter is defined as the quotient of the group of 3-cocycles by the group of 3-coboundaries.) This (and the discussion in point (9) of Example 1.11) relates the equivalence classes of monoidal structures on $\mathcal{C}_A^G$ to $H^3(G, A)$. Similarly, one can relate the equivalence classes of monoidal structures on $\mathbf{Vect}_{\Bbbk}^G$ to $H^3(G, \Bbbk^\times)$. For more details, see Section 2.6 in [EGNO15].

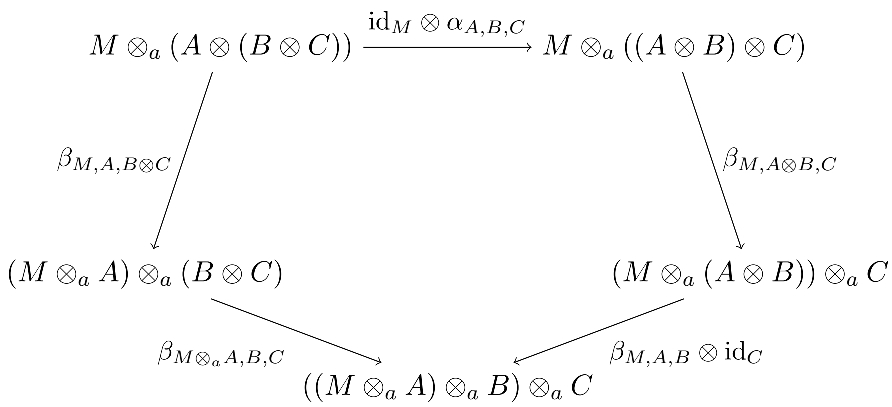
2 Module categories

Unless otherwise stated, we continue to assume in this section that $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is a monoidal category (which we usually abbreviate by $\mathcal{C}$).

Definition 2.1. A *right $\mathcal{C}$-module category* is a quadruple $(\mathcal{M}, \otimes_a, \beta, \vartheta)$, where

- $\mathcal{M}$ is a category,
- $\otimes_a \colon \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ is a bifunctor, called the *action*,
- $\beta\colon\ - \otimes_a (- \otimes -) \to (- \otimes_a -) \otimes_a -$ is a natural isomorphism, called the *associativity constraint*,
- $\vartheta\colon\ - \otimes_a \mathbf{1} \to \mathrm{id}_{\mathcal{M}}$ is a natural isomorphism, called the *unitor*,

subject to the following axioms:
Pentagon axiom: For all objects M of $\mathcal{M}$ and A, B, C of $\mathcal{C}$, the following diagram commutes:



In other words, we have

$$\big(\beta_{M\otimes_a A,B,C}\big) \circ \big(\beta_{M,A,B\otimes C}\big) = \big(\beta_{M,A,B} \otimes \mathrm{id}_C\big) \circ \big(\beta_{M,A\otimes B,C}\big) \circ \big(\mathrm{id}_M \otimes \alpha_{A,B,C}\big).$$

Unit axiom / triangle axiom: For all objects M of $\mathcal{M}$ and A of $\mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccc}
M \otimes_a (\mathbf{1} \otimes A) & \xrightarrow{\beta_{M,\mathbf{1},A}} & (M \otimes_a \mathbf{1}) \otimes_a A \\
& {\scriptstyle \mathrm{id}_M \otimes_a \lambda_A} \searrow \quad \swarrow {\scriptstyle \vartheta_M \otimes_a \mathrm{id}_B} & \\
& M \otimes_a A &
\end{array}
$$

In other words, we have

$$(\vartheta_M \otimes_a \mathrm{id}_A) \circ \beta_{M,\mathbf{1},A} = \mathrm{id}_M \otimes_a \lambda_A.$$

Remark 2.2. (1) We can analogously define a *left $\mathcal{C}$-module category* to be a tuple $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ with an action bifunctor $\otimes_a \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, an associativity constraint

$$\beta \colon (- \otimes -) \otimes_a - \to - \otimes_a (- \otimes_a -)$$

and a unitor $\vartheta \colon \mathbf{1} \otimes_a - \to \mathrm{id}_{\mathcal{M}}$.

(2) Being a module category over $\mathcal{C}$ is not a property of a given category but an additional structure.

Example 2.3. (1) $\mathcal{C}$ is a right $\mathcal{C}$-module category if we set $\otimes_a = \otimes$, $\beta = \alpha$ and $\vartheta = \rho$.

(2) For two categories $\mathcal{C}$ and $\mathcal{D}$, the category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is a right $\mathbf{End}(\mathcal{C})$-module category, where $\otimes_a$ is defined by composition of functors:

$$(F, G) \mapsto F \otimes_a G \coloneqq F \circ G, \qquad (\eta, \nu) \mapsto \eta \otimes_a \nu \coloneqq \eta\nu$$

for functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta \colon F \to F'$ and $\nu \colon G \to G'$.

(3) For a field $\Bbbk$ and a group G, the category $\mathbf{Rep}_{\Bbbk}(G)$ is a right $\mathbf{Vect}_{\Bbbk}(G)$-module category. Indeed, viewing a $\Bbbk$-vector space as a trivial G-module gives rise to a (monoidal) functor $e \colon \mathbf{Vect}_{\Bbbk}(G) \to \mathbf{Rep}_{\Bbbk}(G)$, and we can define $- \otimes_a - = - \otimes e(-)$

Lemma 2.4. *Given a left module category $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ over $\mathcal{C}$, there is a monoidal functor*

$$(F, \varphi, \varepsilon) \colon \mathcal{C} \longrightarrow \mathbf{End}(\mathcal{M})$$

with $F = \mathrm{id}_{\mathcal{C}} \otimes_a -$,

$$\varphi_{A,B} = \beta_{A,B,-} \colon F(A \otimes B) = (A \otimes B) \otimes_a - \xrightarrow{\sim} A \otimes_a (B \otimes_a -) = F(A) \circ F(B)$$

and $\varepsilon = \vartheta \colon \mathbf{1} \otimes_a - \to \mathrm{id}_{\mathcal{M}}$.

Proof. This is straightforward to check using the definitions. □

Lemma 2.5. *A monoidal functor $(F, \varphi, \varepsilon) \colon \mathcal{C} \to \mathbf{End}(\mathcal{M})$ gives rise to a left $\mathcal{C}$-module category structure $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ via $- \otimes_a - = F(-)(-)$,*

$$\beta_{A,B,M} = (\varphi_{A,B})_M \colon (A \otimes B) \otimes_a M = F(A \otimes B)(M) \xrightarrow{\sim} \big(F(A) \circ F(B)\big)(M) = A \otimes_a (B \otimes_a M)$$

and $\vartheta = \varepsilon \colon \mathbf{1} \otimes_a - = F(\mathbf{1})(-) \xrightarrow{\sim} \mathrm{id}_{\mathcal{M}}$.

Remark 2.6. Combining Lemmas 2.4 and 2.5, we see that there is a one-to-one correspondence between left $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors $\mathcal{C} \to \mathbf{End}(\mathcal{M})$.

Definition 2.7. Let $(\mathcal{M}, \otimes_a, \beta, \vartheta)$ and $(\mathcal{M}', \otimes_a', \beta', \vartheta')$ be two right $\mathcal{C}$-module categories. A *right $\mathcal{C}$-module functor* from $\mathcal{M}$ to $\mathcal{M}'$ is a pair (F, γ), where $F\colon \mathcal{M} \to \mathcal{M}'$ is a functor and

$$\gamma\colon F(- \otimes_a -) \longrightarrow F(-) \otimes_a' -$$

is a natural isomorphism such that the following diagrams commute for all objects X, Y of $\mathcal{C}$ and M of $\mathcal{M}$:

$$\begin{array}{ccc}
F\big(M \otimes_a (X \otimes Y)\big) & \xrightarrow{F(\beta_{M,X,Y})} & F\big((M \otimes_a X) \otimes_a Y\big) \\
 & & \downarrow{\scriptstyle \gamma_{M \otimes_a X, Y}} \\
\downarrow{\scriptstyle \gamma_{M, X \otimes Y}} & & F(M \otimes_a X) \otimes_a' Y \\
 & & \downarrow{\scriptstyle \gamma_{M,X} \otimes_a' \mathrm{id}_Y} \\
F(M) \otimes_a' (X \otimes Y) & \xrightarrow[\beta'_{F(M),X,Y}]{} & \big(F(M) \otimes_a' X\big) \otimes_a' Y
\end{array}$$

$$\begin{array}{ccc}
F(M \otimes_a \mathbf{1}) & \xrightarrow{\gamma_{M,\mathbf{1}}} & F(M) \otimes_a' \mathbf{1} \\
{\scriptstyle F(\vartheta_M)} \searrow & & \swarrow {\scriptstyle \vartheta'_{F(M)}} \\
 & F(M) &
\end{array}$$

A *right $\mathcal{C}$-module natural transformation* from a right $\mathcal{C}$-module functor $(F, \gamma)\colon \mathcal{M} \to \mathcal{M}'$ to a right $\mathcal{C}$-module functor $(F', \gamma')\colon \mathcal{M} \to \mathcal{M}'$ is a natural transformation $\psi\colon F \to F'$ such that the following diagram commutes for all objects X of $\mathcal{C}$ and M of $\mathcal{M}$:

$$\begin{array}{ccc}
F(M \otimes_a X) & \xrightarrow{\gamma_{M,X}} & F(M) \otimes_a' X \\
\downarrow{\scriptstyle \psi_{M \otimes_a X}} & & \downarrow{\scriptstyle \psi_M \otimes_a' \mathrm{id}_X} \\
F'(M \otimes_a X) & \xrightarrow[\gamma'_{M,X}]{} & F'(M) \otimes_a' X
\end{array}$$

We write $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{M})$ for the monoidal category of right $\mathcal{C}$-module endofunctors of a right $\mathcal{C}$-module category $\mathcal{M}$, with homomorphisms given by the right $\mathcal{C}$-module natural transformations. The tensor product is defined as the composition of functors. (There is a canonical choice of structure maps which endows the composition of two right $\mathcal{C}$-module functors with the structure of a right $\mathcal{C}$-module functor, cf. part (2) of Remark 1.15)

3 Strictness and coherence

Definition 3.1. A monoidal category $\mathcal{C}$ is called *strict* if

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z \qquad \text{and} \qquad \mathbf{1} \otimes X = X = X \otimes \mathbf{1}$$

for all objects X, Y, Z of $\mathcal{C}$ (note that we require equalities and not isomorphisms) and if all associativity constraints and unitors are identity maps.

Example 3.2. The monoidal category $\mathbf{End}(\mathcal{D})$ of endofunctors of a category $\mathcal{D}$ is strict, and so is the category $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{M})$ of right $\mathcal{C}$-module endofunctors of a right $\mathcal{C}$-module category $\mathcal{M}$.

Theorem 3.3. *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Proof (sketch). We show that $\mathcal{C}$ is equivalent to the strict category $\mathcal{C}' := \mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C})$ of right $\mathcal{C}$-module endofunctors of $\mathcal{C}$. We can define a functor $F\colon \mathcal{C} \to \mathcal{C}'$ by

$$F(A) = (A \otimes - \,,\, \alpha_{A,-,-}) \qquad \text{and} \qquad F(f) = f \otimes -$$

for every object A of $\mathcal{C}$ and every morphism $f\colon A \to B$ in $\mathcal{C}$. A functor $G\colon \mathcal{C}' \to \mathcal{C}$ is given by

$$G(X) = X(\mathbf{1}) \qquad \text{and} \qquad G(\eta) = \eta_{\mathbf{1}}$$

for a right $\mathcal{C}$-module endofunctor (X, γ) of $\mathcal{C}$ and a right $\mathcal{C}$-module natural transformation $\eta\colon X \to Y$. Then one can endow F and G with structure maps to make them monoidal functors, and check that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to the identity functor on $\mathcal{C}'$. Indeed, given a right $\mathcal{C}$-module endofunctor (X, γ) of $\mathcal{C}$ and an object A of $\mathcal{C}$, we have an isomorphism

$$F \circ G\big((X,\gamma)\big)(A) = X(\mathbf{1}) \otimes A \xrightarrow{\gamma^{-1}_{\mathbf{1},A}} X(\mathbf{1} \otimes A) \xrightarrow{X(\lambda_A)} X(A)$$

which gives rise to a right $\mathcal{C}$-module natural isomorphism

$$\varphi_{(X,\gamma)}\colon F \circ G\big((X,\gamma)\big) = \big(X(\mathbf{1}) \otimes -, \alpha_{X(\mathbf{1},-,-)}\big) \longrightarrow (X, \gamma)$$

that is natural in (X, γ). This yields the desired (monoidal) natural isomorphism $F \circ G \to \mathrm{id}_{\mathrm{End}_{\mathrm{end}-\mathcal{C}}(\mathcal{C})}$. The details of the proof are left to the reader. ☐

Example 3.4. In this example, we construct a strict monoidal category that is monoidally equivalent to $\mathbf{Vect}_{\Bbbk}$ for a field $\Bbbk$. Let $\mathbf{Mat}_{\Bbbk}$ be the category whose objects are the natural numbers

$$\mathrm{Ob}(\mathbf{Mat}_{\Bbbk}) := \mathbb{N} = \{0, 1, 2, \ldots\},$$

with homomorphisms from $m \in \mathbb{N}$ to $n \in \mathbb{N}$ given by the set

$$\mathrm{Hom}_{\mathbf{Mat}_{\Bbbk}}(m, n) := \mathrm{Mat}_{n \times m}(\Bbbk)$$

of $n \times m$-matrices over $\Bbbk$ (i.e. matrices with m columns and n rows). Composition is given by matrix multiplication. We can define a monoidal structure on **Mat** by $m \otimes n = m \cdot n$ for $m, n \in \mathbb{N}$ at the level of objects. The tensor product of two matrices $A = (a_{ij}) \in \mathrm{Mat}_{n \times m}(\Bbbk)$ and $B = (b_{ij}) \in \mathrm{Mat}_{n' \times m'}(\Bbbk)$ is the *Kronecker product*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix} \in \mathrm{Mat}_{(nn') \times (mm')}(\Bbbk).$$

and the unit object is $1 \in \mathbb{N}$. (It is straightforward to verify that the Kronecker product is associative.)

We can define a functor $F\colon \mathbf{Mat}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ by $F(m) = \Bbbk^{\oplus m}$ for $m \in \mathbb{N}$, at the level of objects. For $A \in \mathrm{Mat}_{n \times m}(\Bbbk)$, we define $F(A) \in \mathrm{Hom}_{\Bbbk}(\Bbbk^{\oplus m}, \Bbbk^{\oplus n})$ to be left multiplication by A on the column vector space $\Bbbk^{\oplus m}$. It is straightforward to see that F is fully faithful and essentially surjective, hence an equivalence of categories, and that F is monoidal.

Theorem 3.5. *Let $A_1, \ldots, A_n$ be objects of $\mathcal{C}$ and let P and Q be two paranthesized tensor products of $A_1, \ldots, A_n$ (in this order, but not necessarily with the same parenthesization), possibly with copies of the unit object $\mathbf{1}$ inserted in different places. Let $f\colon P \to Q$ and $g\colon P \to Q$ be two isomorphisms that are obtained by composing tensor products of identity morphisms, associativity constraints, unitors and their respective inverses. Then $f = g$.*

Proof. This follows from the strictness theorem. More specifically, let $F\colon \mathcal{C} \to \mathcal{C}_0$ be a monoidal equivalence from $\mathcal{C}$ to a strict monoidal category $\mathcal{C}_0$. Then we have $F(P) = F(Q)$ and $F(f) = F(g)$. Since F induces a bijection between $\mathrm{Hom}_{\mathcal{C}}(P, Q)$ and $\mathrm{Hom}_{\mathcal{C}_0}\big(F(P), F(Q)\big)$, it follows that $f = g$. $\square$

Remark 3.6. An example of a setting in which we can use the coherence theorem 3.5 is Lemma 1.13: For objects A and B of $\mathcal{C}$, the homomorphisms

$$\rho_{A\otimes B} \circ \alpha_{A,B,\mathbf{1}} \colon A \otimes (B \otimes \mathbf{1}) \longrightarrow A \otimes B \qquad \text{and} \qquad \mathrm{id}_A \otimes \rho_B \colon A \otimes (B \otimes \mathbf{1}) \longrightarrow A \otimes B$$

coincide. This does not yield an alternative proof of Lemma 1.13 because the lemma is used in the proof of the strictness theorem 3.3, which is used in turn in the proof of the coherence theorem 3.5.

Another example is that for objects A, B, C, D, E of $\mathcal{C}$, all of the isomorphism between

$$A \otimes (B \otimes (C \otimes (D \otimes E))) \qquad \text{and} \qquad (((A \otimes B) \otimes C) \otimes D) \otimes E$$

that are obtained as compositions of tensor products of identity homomorphisms and associativity constraints coincide.

4 Duals and rigidity

We continue to assume that $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is a monoidal category. In order to simplify the notation, we will often ignore associativity constraints and unitors in the following.

In this section, we want to discuss the notion of dual objects in monoidal categories. We start with a motivating example.

Example 4.1. Let $\Bbbk$ be a field, let V be a finite-dimensional $\Bbbk$-vector space and let $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ be the dual space of V. Then there is a canonical evaluation map

$$\mathrm{ev}_V \colon V^* \otimes V \longrightarrow \Bbbk \qquad \text{with} \quad \xi \otimes v \longmapsto \xi(v).$$

Furthermore, for a basis $\{v_1, \ldots, v_n\}$ of V with dual basis $\{v_1^*, \ldots, v_n^*\}$ of V^* (defined via $v_i^*(v_j) = \delta_{ij}$), there is a coevaluation map

$$\mathrm{coev}_V \colon \Bbbk \longrightarrow V \otimes V^*, \qquad \lambda \longmapsto \lambda \cdot \sum_{i=1}^{n} v_i \otimes v_i^*,$$

and this map is independent of the choice of basis. Using the definitions, one easily checks that the composition

$$V \cong \Bbbk \otimes V \xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V \otimes \Bbbk \cong V$$

coincides with the identity map on V, that is $(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.

Now suppose that we are given a vector space W with linear maps

$$e \colon W \otimes V \longrightarrow \Bbbk \qquad \text{and} \qquad c \colon \Bbbk \longrightarrow V \otimes W.$$

Then we can define linear maps $f \colon W \to V^*$ and $g \colon V^* \to W$ via

$$f(w) = e(w \otimes -) \qquad \text{and} \qquad g(\xi) = (\xi \otimes \mathrm{id}_W) \circ c(1),$$

for $w \in W$ and $\xi \in V^*$. Now one can further show that f and g are mutually inverse if and only if

$$(\mathrm{id}_V \otimes e) \circ (c \otimes \mathrm{id}_V) = \mathrm{id}_V \qquad \text{and} \qquad (e \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes c) = \mathrm{id}_W.$$

This motivates the definition of duals below.

Definition 4.2. A *left dual* of an object X of $\mathcal{C}$ is an object X^* of $\mathcal{C}$ together with homomorphisms

$$\mathrm{ev}_X \colon X^* \otimes X \longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}_X \colon \mathbf{1} \longrightarrow X \otimes X^*,$$

called *evaluation* and *coevaluation*, such that the compositions

$$X \cong \mathbf{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathbf{1} \cong X$$

and

$$X^* \cong X^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} \mathbf{1} \otimes X^* \cong X^*$$

afford the identity homomorphisms on X and X^*, respectively. The equalities

$$\mathrm{id}_X = (\mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \qquad \text{and} \qquad \mathrm{id}_{X^*} = (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X)$$

are called the *zig-zag relations*.

A *right dual* of X is an object ${}^*X$ of $\mathcal{C}$ together with homomorphisms

$$\mathrm{ev}'_X \colon X \otimes X^* \longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}'_X \colon \mathbf{1} \longrightarrow X^* \otimes X,$$

called *evaluation* and *coevaluation*, such that the compositions

$$X \cong X \otimes \mathbf{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} \mathbf{1} \otimes X \cong X$$

and

$$X^* \cong \mathbf{1} \otimes X^* \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{X^*}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{ev}'_X} X^* \otimes \mathbf{1} \cong X^*$$

afford the identity homomorphisms on X and X^*, respectively.

Example 4.3. (1) The dual space V^* of a finite-dimensional vector space V is a left (and right) dual of V in $\mathbf{Vect}_{\Bbbk}$.

(2) For a G-module M in $\mathbf{Rep}_{\Bbbk}(G)$, the dual space M^* becomes a G-module via $(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$. The linear maps $\mathrm{ev}_M \colon M^* \otimes M \to \Bbbk$ and $\mathrm{coev}_M \colon \Bbbk \to M \otimes M^*$ are homomorphisms of G-modules, so M^* is a left (and right) dual of M in $\mathbf{Rep}_{\Bbbk}(G)$.

(3) Let G be a monoid. If for $g \in G$, the one-dimensional graded vector space $\Bbbk_g$ has a left dual then g has an inverse in G, since a tensor product $\big(\bigoplus_h V_h\big) \otimes \Bbbk_g$ admits a non-zero homomorphism to $\Bbbk_e$ only if $hg = e$ for some $h \in G$. If G is a group then every G-graded vector space $V = \bigoplus_g V_g$ has a left (and right) dual, given by the dual space V^* with grading defined by $V_g^* = (V_{g^{-1}})^*$, for $g \in G$.

Lemma 4.4. *If an object X of $\mathcal{C}$ admits a left (or right) dual then the latter is unique up to isomorphism.*

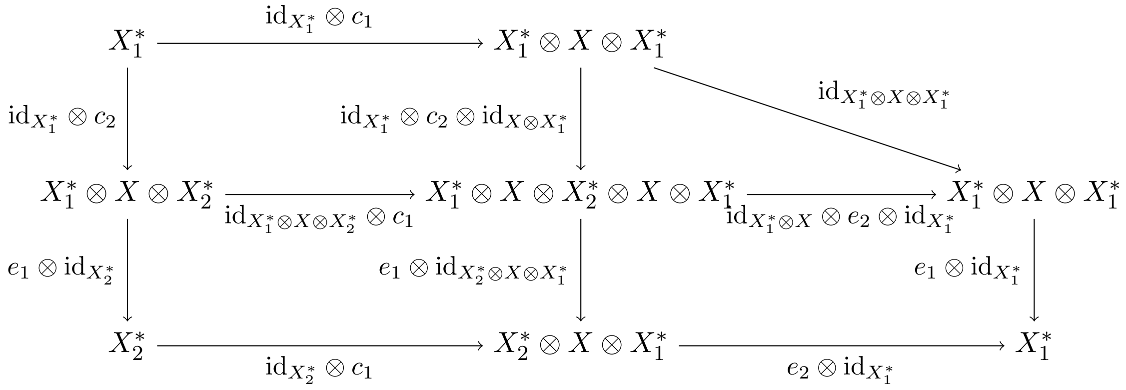
Proof. Let X_1^* and X_2^* be two left dual objects of X and denote by e_1, e_2, c_1, c_2 the corresponding evaluation and coevaluation homomorphisms. We define two homomorphisms

$$f \colon X_1^* \cong X_1^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X_1^*} \otimes c_2} X_1^* \otimes X \otimes X_2^* \xrightarrow{e_1 \otimes \mathrm{id}_{X_2^*}} \mathbf{1} \otimes X_2^* \cong X_2^*$$

and

$$g \colon X_2^* \cong X_2^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X_2^*} \otimes c_1} X_2^* \otimes X \otimes X_1^* \xrightarrow{e_2 \otimes \mathrm{id}_{X^*}} \mathbf{1} \otimes X_1^* \cong X_1^*$$

and consider the following commutative diagram:



The squares commute by the bifunctoriality of the tensor product and the triangle commutes because X_2^* is a left dual of X. Hence the composition along the top right boundary of the diagram coincides with the composition along the bottom left boundary. The former is the identity on X_1^* because X_1^* is a left dual of X, and the latter equals $g \circ f$. Analogously, one sees that $\mathrm{id}^*_{X_2} = f \circ g$, and the claim follows. □

Remark 4.5. More specifically, the left dual of an object of $\mathcal{C}$ is unique up to a unique isomorphism in the following sense: In the notation of Lemma 4.4, assume that we have an homomorphism $h \colon X_1^* \to X_2^*$ such that $e_1 = e_2 \circ (h \otimes \mathrm{id}_X)$ and $c_2 = (\mathrm{id}_X \otimes h) \circ c_1$. Then one can show that h coincides with the isomorphism $f = (e_1 \otimes \mathrm{id}_{X_2^*}) \circ (\mathrm{id}_{X_1^*} \otimes c_2)$ from the proof of Lemma 4.4.

Lemma 4.6. *If an object X of $\mathcal{C}$ has a left dual X^* then X is a right dual of X^*, that is ${}^*(X^*) = X$. Analogously, if X has a right dual then $({}^*X)^* = X$.*

Proof. For the first claim, set $\mathrm{ev}'_{X^*} = \mathrm{ev}_X$ and $\mathrm{coev}'_{X^*} = \mathrm{coev}_X$. The second claim is analogous. □

Lemma 4.7. *Let X and Y be objects of $\mathcal{C}$ with left duals. Then $Y^* \otimes X^*$ is a left dual of $X \otimes Y$. Similarly, if X and Y have right duals then ${}^*Y \otimes {}^*X$ is a right dual of $X \otimes Y$.*

Proof. We define

$$\mathrm{ev}_{X \otimes Y} \colon Y^* \otimes X^* \otimes X \otimes Y \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^*}} Y^* \otimes \mathbf{1} \otimes Y \cong Y^* \otimes Y \xrightarrow{\mathrm{ev}_Y} \mathbf{1}$$

and

$$\mathrm{coev}_{X \otimes Y} \colon \mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \cong X \otimes \mathbf{1} \otimes X^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*}} X \otimes Y \otimes Y^* \otimes X^*,$$

omitting associativity constraints and unitors. It is straightforward to check that these homomorphisms satisfy the zig-zag relations. □

Lemma 4.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $(F, \varphi, \varepsilon) \colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor. If an object X of $\mathcal{C}$ has a left dual X^* then $F(X^*)$ is a left dual of $F(X)$.*

Proof. Consider the homomorphisms

$$\mathrm{ev}_{F(X)} \colon F(X^*) \otimes F(X) \xrightarrow{\varphi^{-1}_{X^*,X}} F(X \otimes X^*) \xrightarrow{F(\mathrm{ev}_X)} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\varepsilon} \mathbf{1}_{\mathcal{D}}$$

and

$$\mathrm{coev}_{F(X)} \colon \mathbf{1}_{\mathcal{D}} \xrightarrow{\varepsilon^{-1}} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{F(\mathrm{coev}_X)} F(X \otimes X^*) \xrightarrow{\varphi_{X,X^*}} F(X) \otimes F(X^*).$$

It is straightforward to verify that $\mathrm{ev}_{F(X)}$ and $\mathrm{coev}_{F(X)}$ satisfy the zig-zag relations, using the zig-zag relations for ev_X and coev_X. □

Definition 4.9. A functor $G\colon \mathcal{D} \to \mathcal{C}$ is called *right adjoint* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ if there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \cong \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big)$$

of functors from $\mathcal{C} \times \mathcal{D}$ to **Set**. In that case, we also say that F is *left adjoint* to G and write $F \dashv G$.

Remark 4.10. We have $F \dashv G$ if and only if there exist natural transformations

$$\varepsilon : FG \longrightarrow \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad \eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow GF,$$

called the *unit* and the *counit* of the adjunction $F \dashv G$, such that the compositions

$$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F\, \eta} FGF \xrightarrow{\varepsilon\, \mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F \qquad \text{and} \qquad G = \mathrm{id}_{\mathcal{C}} \circ G \xrightarrow{\eta\, \mathrm{id}_G} GFG \xrightarrow{\mathrm{id}_G\, \varepsilon} G \circ \mathrm{id}_{\mathcal{D}} = G$$

are equal to id_F and id_G, respectively. The equalities $\mathrm{id}_F = (\varepsilon\, \mathrm{id}_F) \circ (\mathrm{id}_F\, \eta)$ and $\mathrm{id}_G = (\mathrm{id}_G\, \varepsilon) \circ (\eta\, \mathrm{id}_G)$ are called the *zig-zag relations.*

Given a natural isomorphism

$$\psi\colon \mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big),$$

we can define

$$\varepsilon_X = \psi^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})\colon X \longrightarrow GF(X) \qquad \text{and} \qquad \eta_Y = \psi_{G(Y),Y}(\mathrm{id}_{G(Y)})\colon FG(Y) \longrightarrow Y$$

for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$. Conversely, given a unit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and a counit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$, we obtain a natural isomorphism

$$\psi\colon \mathrm{Hom}_{\mathcal{C}}\big(-, G(-)\big) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), -\big)$$

via

$$\psi_{X,Y}(f) = \varepsilon_X \circ F(f)\colon F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\varepsilon_X} Y$$

for objects X of $\mathcal{C}$ and Y of $\mathcal{D}$ and a homomorphism $f\colon X \to G(Y)$ in $\mathcal{D}$.

Example 4.11. In the monoidal category $\mathbf{End}(\mathcal{D})$ of endofunctors of a category $\mathcal{D}$, a left dual of a functor $F\colon \mathcal{D} \to \mathcal{D}$ is the same as a left adjoint of F, and a right dual is the same as a right adjoint.

Lemma 4.12. *If an object X of $\mathcal{C}$ has a left dual X^* then there are adjunctions*

$$- \otimes X \dashv - \otimes X^* \qquad \textit{and} \qquad X^* \otimes - \dashv X \otimes -.$$

In particular, for objects Y and Z of $\mathcal{C}$, there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(Y \otimes X, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z \otimes X^*) \qquad \textit{and} \qquad \mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes Z)$$

which are natural in Y and Z.

Proof. The unit and the counit of the adjunction $- \otimes X \dashv - \otimes X^*$ are given by

$$\varepsilon_Y\colon (Y \otimes X^*) \otimes X \xrightarrow{\alpha_{Y,X^*,X}} Y \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_X} Y$$

and

$$\eta_Y\colon Y \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_Y} (X \otimes X^*) \otimes Y \xrightarrow{\alpha^{-1}_{X,X^*,Y}} X \otimes (X^* \otimes Y).$$

The zig-zag relations for ε and η follow from the zig-zag-relations for the evaluation and coevaluation, and the first isomorphism between Hom-sets is immediate from Remark 4.10. The second adjunction and isomorphism of Hom-sets are obtained analogously. $\square$

Example 4.13. Consider the category **AbGrp** $= \mathbb{Z} - $ **Mod** of abelian groups (or $\mathbb{Z}$-modules), with the usual tensor product $\otimes = \otimes_{\mathbb{Z}}$ over $\mathbb{Z}$ and the unit object $\mathbb{Z}$. Observe that for any two abelian groups A and B, the set $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ can be considered as an abelian group via pointwise addition, and that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \cong A$ (by evaluation at $1 \in \mathbb{Z}$). If the group $A = \mathbb{Z}/3\mathbb{Z}$ has a left dual A^* then

$$A^* \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A^*) \cong \mathrm{Hom}(A, \mathbb{Z}) = 0$$

by Lemma 4.12, contradicting the zig zag relations. Hence $\mathbb{Z}/3\mathbb{Z}$ does not admit a left dual in **AbGrp**.

Remark 4.14. Before we continue discussing duals, some reminders about category theory are in order. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is called *faithful* (or *full*) if the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{f \mapsto F(f)} \mathrm{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big)$$

is injective (respectively surjective). The functor F is called *fully faithful* if it is full and faithful, and it is called *essentially surjective* if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ such that $F(X) \cong Y$. One can show that a functor is fully faithful and essentially surjective if and only if it is an equivalence.

Now for every object X of $\mathcal{C}$, we have a functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \colon \mathcal{C} \longrightarrow \mathbf{Set}$$

and for a homomorphism $f \colon X \to Y$ in $\mathcal{C}$, there is a natural transformation

$$\mathrm{Hom}(f, -) \colon \mathrm{Hom}_{\mathcal{C}}(Y, -) \to \mathrm{Hom}_{\mathcal{C}}(Y, -)$$

with components

$$\mathrm{Hom}(f, Z) \colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \qquad g \mapsto g \circ f.$$

These data give rise to a functor

$$\mathcal{C}^{\mathrm{rev}} \longrightarrow \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$$

which is fully faithful by Yoneda's lemma. In particular, given two objects X and Y of $\mathcal{C}$ such that the functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ are naturally isomorphic, there exists an isomorphism $X \cong Y$ in $\mathcal{C}$.

Remark 4.15. Using Lemma 4.12, we can give an alternative proof of Lemma 4.4: For an element X of $\mathcal{C}$ with two left duals X_1^* and X_2^*, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X_1^*, -) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) \cong \mathrm{Hom}_{\mathcal{C}}(X_2^*, -),$$

and Yoneda's lemma implies that $X_1^* \cong X_2^*$.

Definition 4.16. Assume that X and Y are objects of $\mathcal{C}$ that have left duals and let $f \colon X \to Y$ be a homomorphism. The left dual of f is the homomorphism

$$f^* \colon Y^* \xrightarrow{\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} Y^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{Y^*, X, X^*}} (Y^* \otimes X) \otimes X^* \xrightarrow{\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}} (Y^* \otimes Y) \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} X^*$$

If X and Y have right duals then one similarly defines the right dual ${}^*f \colon {}^*Y \to {}^*X$.

Remark 4.17. Assume that X and Y are objects of $\mathcal{C}$ with left duals, and let $f \colon X \to Y$ be a homomorphism. Then there is a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(X^*, -) & \xrightarrow{\;g \mapsto g \circ f^*\;} & \mathrm{Hom}_{\mathcal{C}}(Y^*, -) \\
\Big\downarrow \wr & & \Big\downarrow \wr \\
\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) & \xrightarrow[\;g \mapsto (f \otimes \mathrm{id}) \circ g\;]{} & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -)
\end{array}$$

where the vertical arrows are given by Remark 4.15. By Yoneda's lemma, f^* is the unique homomorphism from Y^* to X^* that makes this diagram commute.

Lemma 4.18. *Let X, Y and Z be objects of $\mathcal{C}$ that have right duals. For homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$, we have $(g \circ f)^* = f^* \circ g^*$.*

Proof. This follows from Remark 4.17 and the commutativity of the diagram

$$\begin{array}{ccccc}
 & & \overset{h \mapsto h \circ (f^* \circ g^*)}{\curvearrowright} & & \\
\mathrm{Hom}_{\mathcal{C}}(X^*, -) & \xrightarrow{\;h \mapsto h \circ f^*\;} & \mathrm{Hom}_{\mathcal{C}}(Y^*, -) & \xrightarrow{\;h \mapsto h \circ g^*\;} & \mathrm{Hom}_{\mathcal{C}}(Z^*, -) \\
\Big\downarrow \wr & & \Big\downarrow \wr & & \Big\downarrow \wr \\
\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes -) & \xrightarrow[\;h \mapsto (f \otimes \mathrm{id}) \circ h\;]{} & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -) & \xrightarrow[\;h \mapsto (g \otimes \mathrm{id}) \circ h\;]{} & \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes -) \\
 & & \underset{h \mapsto \big((g \circ f) \otimes \mathrm{id}\big) \circ h}{\curvearrowright} & &
\end{array}$$

where the vertical arrows are given by Remark 4.15. ☐

Corollary 4.19. *If every object X in $\mathcal{C}$ has a left dual X^* then there is a contravariant left duality functor $(-)^*\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$ with $X \mapsto X^*$ and $f \mapsto f^*$ for all objects X and all homomorphisms f in $\mathcal{C}$. Analogously, if every object X in $\mathcal{C}$ has a right dual ${}^*X$ then there is a right duality functor ${}^*(-)\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$.*

Proof. This follows from Lemmas 4.4 and 4.18. ☐

Remark 4.20. Suppose that all objects in $\mathcal{C}$ have right duals. Then the right duality functor canonically defines a monoidal functor $\big((-)^*, \varphi, \varepsilon\big)\colon \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}^{\mathrm{op}}$. This follows from Lemmas 4.4 and 4.7 and the fact that $(f \otimes g)^* = g^* \otimes f^*$ for homomorphisms f and g in $\mathcal{C}$. (This fact can be proven by arguing as in Lemma 4.18.)

Definition 4.21. A monoidal category $\mathcal{C}$ is called *rigid* if every object X has a right dual X^* and a left dual ${}^*X$.

Remark 4.22. In a rigid monoidal category, the left duality functor is an equivalence and its quasi-inverse is the right duality functor by Lemma 4.6.

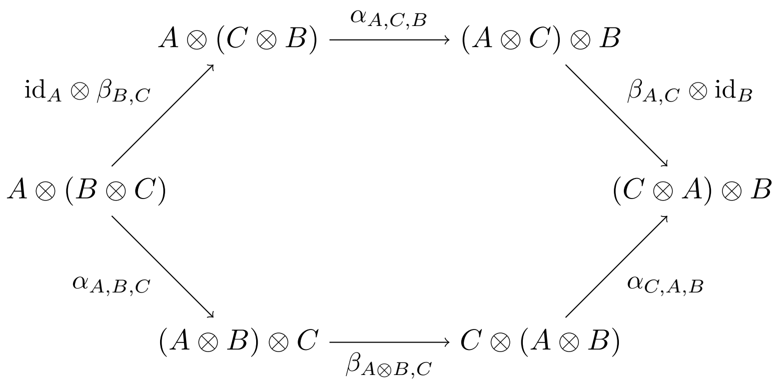
The name *rigid* is justified by the following lemma:

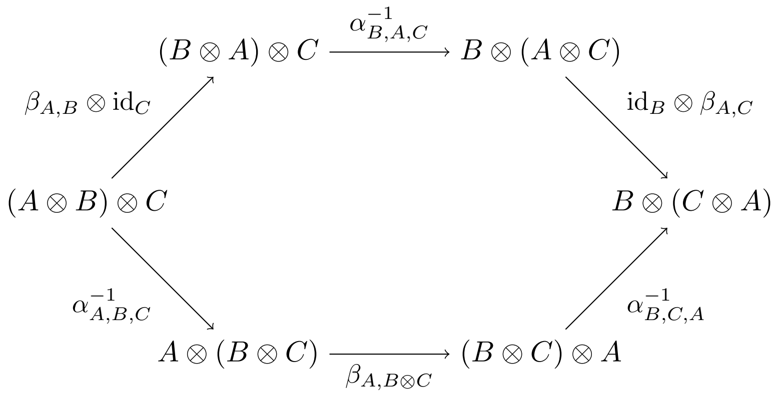
Lemma 4.23. *Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories and let $(F, \varphi, \varepsilon)$ and $(G, \varphi', \varepsilon')$ be monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. Further let $u\colon F \to G$ be a monoidal natural transformation. Then u is a natural isomorphism.*

Proof. The proof will be given in the exercises. ☐

5 Braided monoidal categories

Definition 5.1. A *braiding* on $\mathcal{C}$ is a natural isomorphism $\beta\colon -\otimes- \to -\otimes^{\mathrm{op}}-$ (i.e. with components $\beta_{A,B}\colon A\otimes B \to B\otimes A$) that satisfies the *hexagon axiom*, that is, such that the following diagrams commute for all objects A, B, C of $\mathcal{C}$:





A *braided monoidal category* is a pair $(\mathcal{C}, \beta)$, where $\mathcal{C}$ is a monoidal category and β is a braiding.

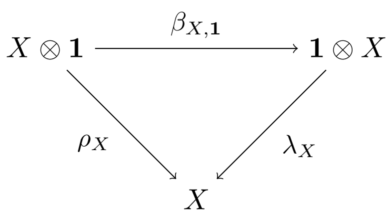
Example 5.2. (1) The category **Set** with the monoidal structure by Cartesian product admits a braiding β with $\beta_{X,Y}\colon X\times Y \to Y\otimes X$ given by $(x,y)\mapsto(y,x)$ for sets X and Y.

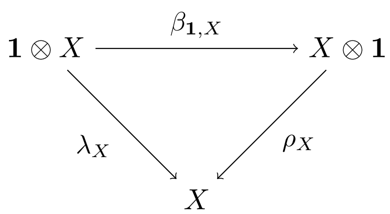
(2) The category $\Bbbk - \mathbf{Vect}$ of vector spaces over a field $\Bbbk$ with the ordinary tensor product has a braiding β with $\beta_{X,Y}\colon X\otimes Y \to Y\otimes X$ determined by $x\otimes y\mapsto y\otimes x$ for $\Bbbk$-vector spaces X and Y and $x\in X$, $y\in Y$.

Lemma 5.3. *In a braided monoidal category* $(\mathcal{C}, \beta)$*, we have*

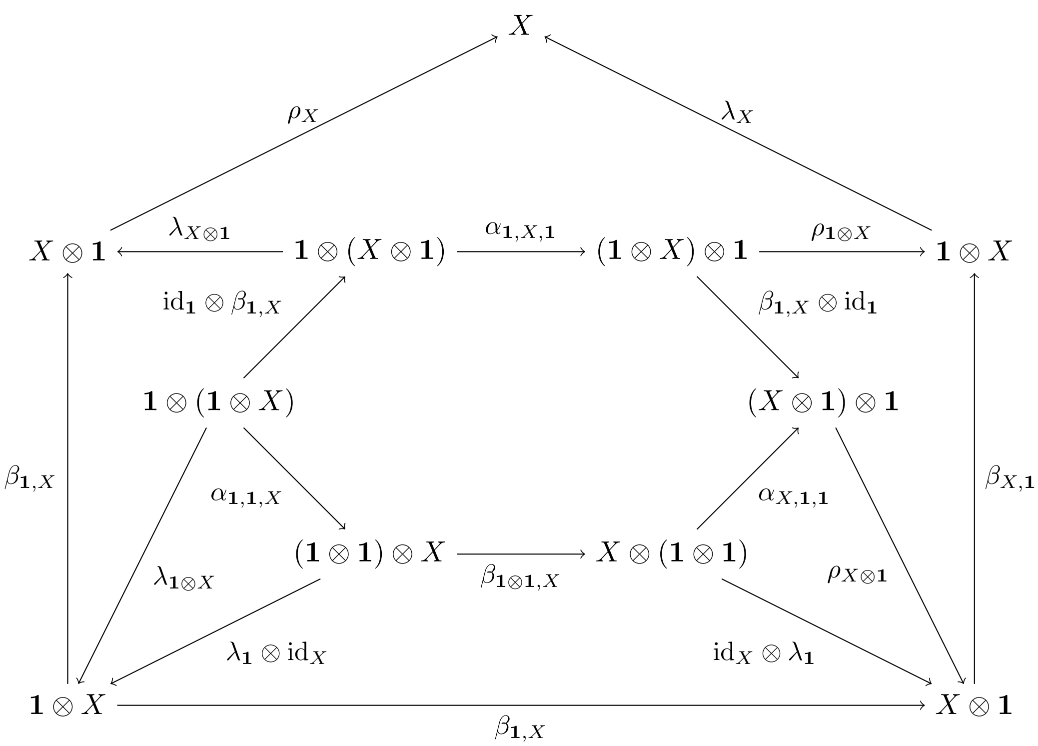
$$\lambda_X = \rho_X \circ \beta_{\mathbf{1},X}, \qquad \rho_X = \lambda_X \circ \beta_{X,\mathbf{1}} \qquad \textit{and} \qquad \beta_{\mathbf{1},X} = \beta^{-1}_{X,\mathbf{1}}$$

for all objects X of $\mathcal{C}$. In other words, the following diagrams commute:





Proof. Using the hexagon axiom, the naturality of β and the unitors, and the coherence theorem (see Theorem 3.5), we get the following commutative diagram:



This implies that we have

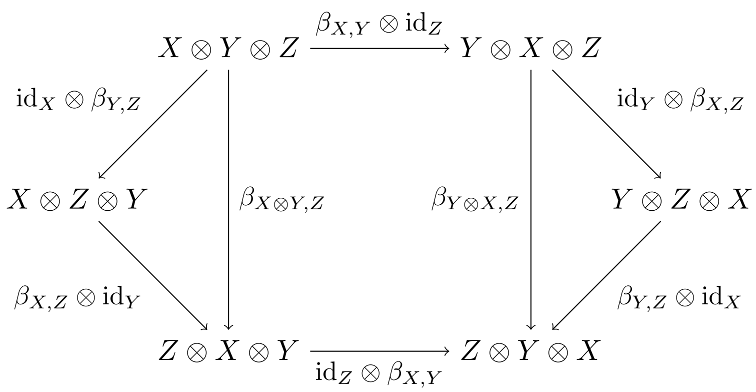
$$\beta_{1,X} = \beta_{X,1} \circ \lambda_X^{-1} \circ \rho_X \circ \beta_{1,X}$$

and therefore $\lambda_X \circ \beta_{X,\mathbf{1}}^{-1} = \rho_X$. The equations $\rho_X \circ \beta_{X,\mathbf{1}} = \lambda_X$ and $\beta_{\mathbf{1},X} = \beta_{X,\mathbf{1}}^{-1}$ can be proven analogously. ☐

Lemma 5.4. *Let $\mathcal{C}$ be a strict monoidal category and let β be a braiding on $\mathcal{C}$. Then $\mathcal{C}$ satisfies the* braid relations*, that is, for objects X, Y, Z of $\mathcal{C}$, we have*

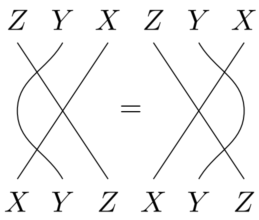
$$(\beta_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes \beta_{Y,X}) \circ (\beta_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \beta_{Y,Z}).$$

Proof. Since $\mathcal{C}$ is strict, the diagrams in the hexagon axiom become triangles, and by naturality of the braiding, we obtain the following commutative diagram:



The braid relation can be read off along the perimeter of the diagram. ☐

Remark 5.5. The braid relations can be depicted by the following diagram:



Remark 5.6. Recall from Theorem 3.3 that every monoidal category is equivalent to a strict monoidal category. This (or a more elaborate version of the argument in the proof of Lemma 5.4) can be used to prove a version of the braid relations for non-strict monoidal categories.

Definition 5.7. A braiding β on $\mathcal{C}$ is called *symmetric* if it satisfies $\beta_{Y,X} \circ \beta_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all objects X, Y of $\mathcal{C}$. A *symmetric monoidal category* is a braided monoidal category with a symmetric braiding β.

Example 5.8. Let G be a group and A an abelian group, and suppose that $\mathcal{C}_A^G$ admits a braiding. Then we have

$$\delta_{gh} = \delta_g \otimes \delta_h \cong \delta_h \otimes \delta_g \cong \delta_{hg}$$

for all $g, h \in G$, so $gh = hg$ and G is abelian. A braiding $\beta\colon -\otimes- \to -\otimes^{\mathrm{op}}-$ on $\mathcal{C}_A^G$ is the same as a collection

$$A \quad \ni \quad \beta_{g,h}\colon \delta_{gh} = \delta_g \otimes \delta_h \longrightarrow \delta_h \otimes \delta_g \cong \delta_{hg}$$

satisfying the hexagon axiom, i.e. a map $\beta\colon G \times G \to A$ such that

$$\beta(g,k)\cdot\beta(h,k) = \beta(gh,k) \qquad \text{and} \qquad \beta(g,h)\cdot\beta(g,k) = \beta(g,hk)$$

for $g, h, k \in G$. In other words, a braiding on $\mathcal{C}_A^G$ is the same as a $\mathbb{Z}$-bilinear map $\beta\colon G \times G \to A$. The braiding is symmetric if and only if $\beta(g,h)\cdot\beta(h,g) = e$ for all $g, h \in G$.

For a field $\Bbbk$ one similarly finds that $\mathbf{Vect}_\Bbbk^G$ admits a braiding if and only if G is abelian, and that every braiding corresponds to a choice of $\mathbb{Z}$-bilinear map $G \times G \longrightarrow \Bbbk^\times$.

Now suppose that $G = \mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$ is the cyclic group of order n. For an n-th root of unity $\zeta \in \Bbbk^\times$, we can define a biliear map $\beta\colon G \times G \longrightarrow \Bbbk^\times$ via

$$\beta(\overline{a}, \overline{b}) = \zeta^{\overline{a\cdot b}}.$$

For $n = 2$ and $\zeta = -1$, the resulting braided monoidal category

$$\mathbf{SVect}_\Bbbk = \left(\mathbf{Vect}_\Bbbk^{\mathbb{Z}/2\mathbb{Z}}, \beta\right)$$

is called the category of *super $\Bbbk$-vector spaces*. It is straightforward to see that $\mathbf{SVect}_\Bbbk$ is a symmetric monoidal category.

Definition 5.9. A *braided monoidal functor* between braided monoidal categories $(\mathcal{C}, \beta)$ and $(\mathcal{D}, \beta')$ is a monoidal functor $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ such that for all objects X, Y of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
F(X\otimes Y) & \xrightarrow{\gamma_{X,Y}} & F(X)\otimes F(Y) \\
{\scriptstyle F(\beta_{X,Y})}\big\downarrow & & \big\downarrow{\scriptstyle \beta'_{F(X),F(Y)}} \\
F(Y\otimes F(X)) & \xrightarrow[\gamma_{Y,X}]{} & F(Y)\otimes F(X)
\end{array}$$

If $(\mathcal{C}, \beta)$ and $(\mathcal{D}, \beta')$ are symmetric then we call $(F, \varphi, \varepsilon)$ a *symmetric monoidal functor*.

Intermission

At this point, let us pause for a while to contemplate what we have learned so far and explain the contents of the following sections. Up to this point, we have seen the definitions of monoidal categories, duals and braidings and studied some of their most important properties (strictness and coherence, behavior of duals under monoidal functors, braid relations, etc...). One key example of a monoidal category that is both rigid and braided is the category $\mathbf{Rep}_{\Bbbk}(G)$ of representations of a field $\Bbbk$. For the rest of this course, we will essentially be concerned with the question what hypotheses we need to impose on a (monoidal) category in order for it to be (monoidally) equivalent to the category of representations of a group, or some other algebraic object. As a first step, let us fix some algebraic object B and list some properties of the category of (finite-dimensional) modules (or representations) of B that would have to be shared by any category that is equivalent to the latter.

(1) The set of homomorphisms between to B-modules has a canonical $\Bbbk$-vector space structure and composition of homomorphisms is $\Bbbk$-linear.

(2) We can form the direct sum of finitely many B-modules, and there is a zero B-module $\{0\}$.

(3) We can consider subobjects and quotients of B-modules, and for a homomorphism of B-modules, the kernel and cokernel are also B-modules.

(4) There is a 'forgetful functor', i.e. a functor that assigns to every representation the underlying vector space.

The points above can be stated more abstractly as saying that any category that is equivalent to a category of representations should be (1) $\Bbbk$-*linear*, (2) *additive*, (3) *abelian* and should (4) admit a *fiber functor*. All of these terms will be defined below, and as we will see later every category $\mathcal{A}$ that satisfies (1)–(4) is equivalent to the category of *comodules* over a *coalgebra* C. (These are dual concepts of the notions of algebras and modules.) If the category $\mathcal{A}$ is also $\Bbbk$-linearly monoidal (a *tensor category*) and F is a monoidal functor then we can endow C with an algebra structure that is compatible with the coalgebra structure, thus making C a *bialgebra*. This gives rise to a bijection between tensor categories with fiber functors (up to equivalence) and bialgebras (up to isomorphism), and every additional structure on the tensor category (e.g. rigidity, braiding) gives rise to an additional structure on the bialgebra (e.g. antipode, R-matrix). In the special case of a rigid symmetric monoidal category, we will see that the extra structure that we get on the bialgebra allows us to define a *group scheme* whose category of representations recovers the abelian category that we started from. Finally, we will turn to $\mathbb{C}$-linear abelian rigid symmetric monoidal categories and discuss a theorem of P. Deligne, which states that a very mild assuption on the growth of tensor powers (with respect to the length of a composition series) is equivalent to the existence of a *super fiber functor* (whose codomain is the category $\mathbf{SVect}_{\mathbb{C}}$ of super $\mathbb{C}$-vector spaces, instead of the category $\mathbf{Vect}_{\mathbb{C}}$). By analogy with the above discussion, this implies that every $\mathbb{C}$-linear abelian rigid symmetric monoidal category which satisfies the aforementioned growth-assumption is monoidally equivalent to a category of representations of a *super group scheme*.

6 Monoids and comonoids

As before, we fix a monoidal category $\mathcal{C}$. We will mostly suppress associativity constraints and unitors from the notation, for the sake of simplicity.

Definition 6.1. (1) A *monoid* in $\mathcal{C}$ is an object M with homomorphisms

$$\mu\colon M \otimes M \longrightarrow M \qquad \text{and} \qquad \eta\colon \mathbf{1} \longrightarrow M,$$

called *multiplication* and *unit*, such that

$$\mu \circ (\mathrm{id}_M \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}_M) \qquad \text{and} \qquad \mu \circ (\mathrm{id}_M \otimes \eta) = \mathrm{id}_M = \mu \circ (\eta \otimes \mathrm{id}_M).$$

A homomorphism between monoids (M, μ, η) and (M', μ', η') is a homomorphism $f\colon M \to M'$ in $\mathcal{C}$ such that $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$.

(2) A *comonoid* in $\mathcal{C}$ is an object C with homomorphisms

$$\delta\colon C \longrightarrow C \otimes C \qquad \text{and} \qquad \varepsilon\colon C \longrightarrow \mathbf{1}$$

called *comultiplication* and *counit*, such that

$$(\mathrm{id}_C \otimes \delta) \circ \delta = (\delta \otimes \mathrm{id}_C) \circ \delta \qquad \text{and} \qquad (\mathrm{id}_C \otimes \varepsilon) \circ \delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C) \circ \delta.$$

A homomorphism between comonoids (C, δ, ε) and $(C', \delta', \varepsilon')$ is a homomorphism $g\colon C \to C'$ in $\mathcal{C}$ such that $\delta' \circ g = (g \otimes g) \circ \delta$ and $\varepsilon' \circ g = \varepsilon$.

Example 6.2. (1) A monoid in the category $\mathbf{Vect}_{\Bbbk}^{\infty}$ of (possibly infinite-dimensional) vector spaces is the same as a $\Bbbk$-algebra. A comonoid in $\mathbf{Vect}_{\Bbbk}^{\infty}$ is called a *coalgebra*. For every finite-dimensional algebra (A, μ, η), the dual space A^* is a coalgebra with comultiplication and counit given by

$$\delta = \mu^*\colon A^* \to (A \otimes A)^* \cong A^* \otimes A^* \qquad \text{and} \qquad \varepsilon = \eta^*\colon A^* \to \Bbbk^* \cong \Bbbk.$$

Similarly, the dual of a finite-dimensional coalgebra has a canonical algebra structure.

(2) Given two $\Bbbk$-algebras A and B (or $\Bbbk$-coalgebras C and D), the tensor product $A \otimes B$ (or $C \otimes D$) over $\Bbbk$ becomes a $\Bbbk$-algebra (or $\Bbbk$-coalgebra) with multiplication (or comultiplication)

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes s_{B,A} \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$

$$C \otimes D \xrightarrow{\delta_C \otimes \delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C \otimes s_{C,D} \otimes \mathrm{id}_D} C \otimes D \otimes C \otimes D,$$

where s denotes the symmetric braiding on $\mathbf{Vect}_{\Bbbk}^{\infty}$. This endows the category $\mathbf{Alg}_{\Bbbk}$ of $\Bbbk$-algebras (or $\mathbf{Coalg}_{\Bbbk}$ of $\Bbbk$-coalgebras) with a monoidal structure. A comonoid in $\mathbf{Alg}_{\Bbbk}$ is called a *bialgebra*. In other words, a bialgebra is an algebra which is also a coalgebra, in such a way that the comultiplication and the counit are homomorphisms of algebras. Equivalently, we could require that the multiplication and unit are homomorphisms of coalgebras, so a bialgebra can also be defined as a monoid in $\mathbf{Coalg}_{\Bbbk}$.

More generally, if $\mathcal{C}$ has a braiding β then we can use β to define tensor products of monoids or comonoids in $\mathcal{C}$. A bimonoid in $\mathcal{C}$ is a comonoid in the category of monoids in $\mathcal{C}$ or (equivalently) a monoid in the category of comonoids in $\mathcal{C}$.

(3) Let **Cat** be the category of small categories (i.e. categories where the objects form a set), and regard **Cat** as a monoidal category, with tensor product defined by the product of categories from Remark 1.5. Then a monoid in **Cat** is the same as a strict monoidal category.

Lemma 6.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathcal{D}$ a monoidal functor and (A, μ, η) a monoid in $\mathcal{C}$. Then $F(A)$ becomes a monoid in $\mathcal{D}$ via*

$$F(A) \otimes F(A) \xrightarrow{\varphi_{A,A}} F(A \otimes A) \xrightarrow{F(\mu)} F(A) \qquad \textit{and} \qquad \mathbf{1}_D \xrightarrow{\varepsilon^{-1}} F(\mathbf{1}_C) \xrightarrow{F(\eta)} F(A).$$

Definition 6.4. (1) A *left module* over a monoid (A, μ, η) in $\mathcal{C}$ is an object M of $\mathcal{C}$ with a homomorphism $a\colon A \otimes M \to M$ such that

$$a \circ (\mu \otimes \mathrm{id}_M) = a \circ (\mathrm{id}_A \otimes a) \qquad \text{and} \qquad a \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M.$$

A homomorphism between left modules (M, a) and (M', a') is a homomorphism $f\colon M \to M'$ in $\mathcal{C}$ such that $f \circ a = a' \circ (f \otimes \mathrm{id}_A)$.

(2) A *left comodule* over a comonoid (C, δ, ε) in $\mathcal{C}$ is an object N of $\mathcal{C}$ with a homomorphism $c\colon N \to C \otimes N$ such that

$$(\delta \otimes \mathrm{id}_N) \circ c = (\mathrm{id}_C \otimes c) \circ c \qquad \text{and} \qquad (\varepsilon \otimes \mathrm{id}_N) \circ c = \mathrm{id}_N.$$

A homomorphism between left comodules (N, c) and (N', c') is a homomorphism $g\colon N \to N'$ in $\mathcal{C}$ such that $(f \otimes \mathrm{id}_C) \circ c = c' \circ f$.

Remark 6.5. For a finite-dimensional algebra A and a finite-dimensional A-module (M, a), the dual space M^* is a comodule over the coalgebra A^*, with coaction given by

$$c = a^*\colon M^* \longrightarrow (A \otimes M)^* \cong A^* \otimes M^*.$$

This gives rise to an equivalence between the opposite category $(A - \mathbf{mod})^{\mathrm{op}}$ of the category of finite-dimensional A-modules and the category $A^* - \mathbf{comod}$ of finite-dimensional A^*-comodules.

Remark 6.6. Let $(B, \mu, \eta, \delta, \varepsilon)$ be a bialgebra and let M and N be B-modules. Then the tensor product $M \otimes N$ over $\Bbbk$ becomes a $B \otimes B$-module via

$$B \otimes B \otimes M \otimes N \xrightarrow{\mathrm{id}_A \otimes s_{B,M} \otimes \mathrm{id}_N} B \otimes M \otimes B \otimes N \xrightarrow{a_M \otimes a_N} M \otimes N,$$

and since the comultiplication $\delta\colon B \to B \otimes B$ is an algebra homomorphism, we can view $M \otimes N$ as a B-module (with $b \cdot (m \otimes n) = \delta(b) \cdot (m \otimes n)$ for $b \in B$, $m \in M$ and $n \in N$). Furthermore, the counit $\varepsilon\colon B \to \Bbbk$ allows us to view the one-dimensional vector space $\Bbbk$ as a B-module, called the *trivial* B-module. This endows the category $B - \mathbf{mod}$ of finite-dimensional B-modules with a monoidal structure. (The associativity constraint is defined using the coassociativity of δ and the unitors are defined using the counitality of ε.) Similarly, we can define a comodule structure on the tensor product of two B-comodules and endow the one-dimensional vector space $\Bbbk$ with a B-comodule structure (via the unit η of B). This endows the category $B - \mathbf{comod}$ of finite dimensional left B-comodules with a monoidal structure.

In the following, we will mostly work with coalgebras and comodules rather than algebras and modules. The reason for this choice is that the former are more well-behaved in a number of ways, as witnessed by the following result (know as the *fundamental theorem of comodules and coalgebras*):

Theorem 6.7. *Let (C, δ, ε) be a coalgebra over $\Bbbk$. Then every C-comodule is the union of its finite-dimensional sub-comodules. Further more, C is the union of its finite-dimensional sub-coalgebras*

Proof. Let N be a C-comodule with coaction map $c\colon N \to C \otimes N$. The first claim follows if we prove that every element $n \in N$ is contained in a finite dimensional sub-comodule of N. Since a choice of bases for C and N affords a basis for $C \otimes N$ (by taking tensor products of basis vectors), there are elements $c_1, \ldots, c_r \in C$ and $n_1, \ldots, n_r \in N$ such that $c(n) = c_1 \otimes n_1 + \cdots + c_r \otimes n_r$. In particular, there is a finite-dimensional subspace $N' = \langle n_1, \ldots, n_r \rangle_\Bbbk$ of N such that $c(n) \in C \otimes N'$. We claim that $M := c^{-1}(C \otimes N')$ is a finite-dimensional sub-comodule of N. Indeed, we have

$$c^{-1}(C \otimes N') \subseteq c^{-1}\big((\varepsilon \otimes \mathrm{id}_N)^{-1}(N')\big) = \big((\varepsilon \otimes \mathrm{id}_N) \circ c\big)^{-1}(N') = N'$$

whence M is finite-dimensional. Furthermore, if we regard $C \otimes N$ as a C-comodule with coaction

$$c' = \delta \otimes \mathrm{id}_V : C \otimes N \longrightarrow C \otimes C \otimes N$$

then the coaction $c\colon N \to C \otimes N$ is a homomorphism of comodules and $C \otimes N'$ is a sub-comodule of $C \otimes N$; hence $M = c^{-1}(C \otimes N')$ is a subcomodule of N.

Next note that C can be regarded as a comodule over $C^{\mathrm{op}} \otimes C$, where C^{op} is the coalgebra with underlying vector space C and comultiplication $\delta^{\mathrm{op}} = s_{C,C} \circ \delta$. Then a $C \otimes C^{\mathrm{op}}$-comodule of C is the same as a sub-coalgebra of C, so the second claim follows from the first. ☐

7 Linear and abelian categories

Definition 7.1. Let R be a commutative ring. An R-linear category is a category such that all Hom-sets are equipped with an R-module structure and composition of homomorphisms is R-bilinear. An R-linear functor between R-linear categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that the maps

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{f \mapsto F(f)} \mathrm{Hom}_{\mathcal{D}}\big(F(X), F(Y)\big)$$

are R-linear, for all objects A and B of $\mathcal{A}$.

Example 7.2. (1) For a commutative ring R, the category $R-\mathbf{Mod}$ is R-linear. In particular, the category $\mathbf{AbGrp} = \mathbb{Z} - \mathbf{Mod}$ of abelian groups (or $\mathbb{Z}$-modules) is $\mathbb{Z}$-linear.

(2) For a field $\Bbbk$ and a group G, the categories $\mathbf{Vect}_{\Bbbk}$, $\mathbf{Vect}_{\Bbbk}^{G}$ and $\mathbf{Rep}_{\Bbbk}(G)$ are $\Bbbk$-linear.

(3) A $\mathbb{Z}$-linear category is also called *pre-additive*.

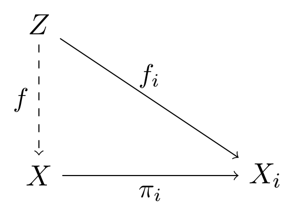
(4) An R-linear category $\mathcal{C}$ with a monoidal structure such that $\otimes$ is R-bilinear on Hom-sets, i.e. such that the maps

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X', Y') \xrightarrow{(f,g) \mapsto f \otimes g} \mathrm{Hom}_{\mathcal{C}}(X \otimes X', Y \otimes Y')$$

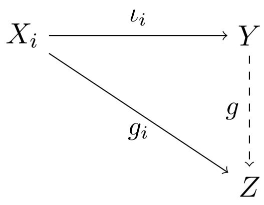
are R-bilinear for all objects X, Y, X', Y' of $\mathcal{C}$, is called *R-linearly monoidal.*

Definition 7.3. Let $(X_i)_i$ be a family of objects of $\mathcal{C}$.

(1) A *product* of $(X_i)_i$, if it exists, is an object $X = \prod_i X_i$ of $\mathcal{C}$ with homomorphisms $\pi_i\colon X \to X_i$ (called *projections*) such that for every object Z of $\mathcal{C}$ with homomorphisms $f_i\colon Z \to X_i$, there exists a unique homomorphism $f\colon Y \to Z$ such that $\pi_i \circ f = f_i$.



(2) A *coproduct* of $(X_i)_i$, if it exists, is an object $Y = \coprod_i X_i$ of $\mathcal{C}$ with homomorphisms $\iota_i\colon X_i \to Y$ (called *inclusions*) such that for every object Z of $\mathcal{C}$ with homomorphisms $g_i\colon X_i \to Z$, there exists a unique homomorphism $g\colon Y \to Z$ such that $g \circ \iota_i = g_i$.



Example 7.4. (1) In the category **Set**, the product is the cartesian product and the coproduct is the disjoint union.

(2) In $\mathbf{Vect}_{\Bbbk}^{\infty}$, the product of $(V_i)_i$ is the direct product (i.e. the set of I-tuples $(v_i)_i$ with $v_i \in V_i$ and pointwise addition and scalar multiplication) and the coproduct is the direct sum (i.e. the subspace of the product consisting of the I-tuples $(v_i)_i$ where v_i is zero for all but finitely many $i \in I$). In particular, finite products and coproducts coincide in $\mathbf{Vect}_{\Bbbk}^{\infty}$ (and in $\mathbf{Vect}_{\Bbbk}$).

Remark 7.5. The universal property of coproducts implies that for any family $(X_i)_i$ such that the coproduct $\coprod_i X_i$ exists and for an object Y of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}\big(\coprod\nolimits_i X_i, Y\big) \cong \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

In particular, in the category $\mathbf{Vect}_{\Bbbk}^{\infty}$, we have canonically

$$\mathrm{Hom}_{\Bbbk}\big(\bigoplus\nolimits_i X_i, Y\big) \cong \prod_i \mathrm{Hom}_{\Bbbk}(X_i, Y) \qquad \text{and} \qquad \big(\bigoplus\nolimits_i X_i\big)^* \cong \prod_i X_i^*.$$

Definition 7.6. A pre-additive category $\mathcal{A}$ is called *additive* if all finite products exist in $\mathcal{A}$.

Example 7.7. The categories **AbGrp**, $\mathbf{Vect}_{\Bbbk}$, $A - \mathbf{mod}$, $\mathbf{Rep}_{\Bbbk}(G)$ and $\mathbf{Vect}_{\Bbbk}^{G}$ are all additive.

Remark 7.8. Let $\mathcal{A}$ be an additive category.

(1) Finite products and coproducts coincide in $\mathcal{A}$. (We could equivalently define an additive category as a pre-additive category having all finite coproducts.)

(2) We can form the empty product (or coproduct) in $\mathcal{A}$, which we call the zero object and denote by 0. Using the universal property of products and coproducts, one sees that for every object A of $\mathcal{A}$, there is a unique homomorphism $0\colon 0 \to A$ and a unique homomorphism $0\colon A \to 0$. For every pair of objects A, B of $\mathcal{A}$, the zero homomorphism in the abelian group $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is the composition $A \to 0 \to B$.

Remark 7.9. While being pre-additive is an extra structure on a category, being additive is a property: For a category $\mathcal{C}$ in which all finite products and coproducts exist, the empty product is a final object and the empty coproduct is an initial object. Suppose that the unique hommorphism from the empty coproduct to the empty product is an isomorphism, and call the resulting initial and final object the zero object 0 of $\mathcal{C}$. Then for all objects X, Y of $\mathcal{C}$ the canonical homomorphisms

$$X \amalg Y \xrightarrow{\mathrm{id}_X \amalg 0} X \amalg 0 \cong X \qquad \text{and} \qquad X \amalg Y \xrightarrow{0 \amalg Y} 0 \amalg Y \cong Y$$

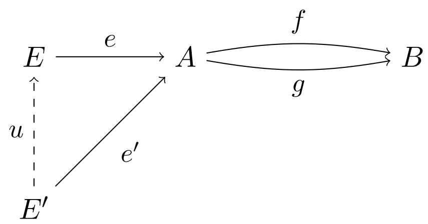
give rise to a homomorphism $X \amalg Y \to X \Pi Y$ via the universal property of the product, and we further assume that the latter is an isomorphism for all X, Y. Then we can define a commutative monoid structure on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as follows: Using the universal properties of product and coproduct, we can define a diagonal embedding $X \to X \Pi X$ and a diagonal projection $Y \Pi Y \cong Y \amalg Y \to Y$, and the sum of two homomorphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is defined as the composition

$$f + g: \quad X \longrightarrow X \Pi X \xrightarrow{f \Pi g} Y \Pi Y \longrightarrow Y.$$

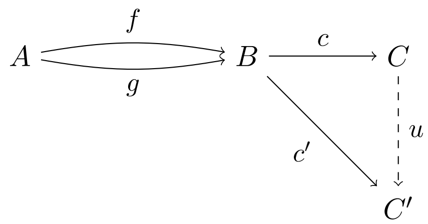
Now one can show that $\mathcal{C}$ is additive if and only if the commutative monoid $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a group for all objects X, Y of $\mathcal{C}$. In summary, a category is additive if and only if all finite products and coproducts exist, the empty product is isomorphic to the empty coproduct, the canonical homomorphism from the coproduct to the product is an isomorphism and every homomorphism has an inverse with respect to the addition on Hom-sets defined above.

Definition 7.10. Let $\mathcal{A}$ be a category, let A and B be objects of $\mathcal{A}$ and let $f\colon A \to B$ and $g\colon A \to B$ be two homomorphisms in $\mathcal{A}$.

(1) The *equalizer* $\mathrm{eq}(f, g) = (E, e)$ of f and g, if it exists, is an object E with a homomorphism $e\colon E \to A$ with $f \circ e = g \circ e$ such that for every object E' of $\mathcal{A}$ and homomorphism $e'\colon E' \to A$ with $f \circ e' = g \circ e'$, there is a unique homomorphism $u\colon E' \to E$ such that $e' = e \circ u$.



(2) The *coequalizer* $\mathrm{coeq}(f, g) = (C, c)$ of f and g, if it exists, is an object C with a homomorphism $c\colon B \to C$ with $c \circ f = c \circ g$ such that for every object C' of $\mathcal{A}$ and homomorphism $c'\colon B \to C'$ with $c' \circ f = c' \circ g$, there is a unique homomorphism $u\colon C \to C'$ such that $c' = u \circ c$.

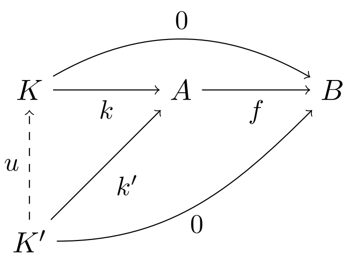


(3) If $\mathcal{A}$ is additive and $f\colon A \to B$ is a homomorphism in $\mathcal{A}$ then the *kernel* (or *cokernel*) of f, if it exists, is the equalizer (or coequalizer)

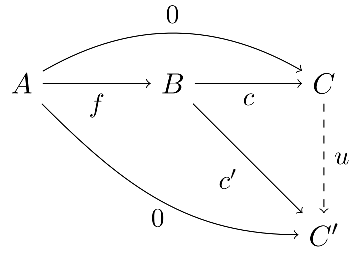
$$\ker(f) = \mathrm{eq}(f, 0), \qquad \mathrm{cok}(f) = \mathrm{coeq}(f, 0)$$

of f and $0\colon A \to B$.

Remark 7.11. Using the universal properties of the equalizer and the coequalizer, it is straightforward to derive the following universal properties of the kernel and the cokernel: The kernel of a homomorphism $f\colon A \to B$ in an additive category $\mathcal{A}$ is an object K with a homomorphism $k\colon K \to A$ such that $f \circ k = 0$, and such that for every object K' with a homomorphism $k'\colon K' \to A$ such that $f \circ k' = 0$, there is a unique homomorphism $u\colon K \to K'$ such that $k' = k \circ u$.



Dually, the cokernel $c\colon Y \to C = \mathrm{cok}(f)$ of f, if it exists, is defined by the universal property displayed in the following diagram:



Definition 7.12. An additive category is called *pre-abelian* if every homomorphism has a kernel and a cokernel.

Definition 7.13. A homomorphism $f\colon X \to Y$ in a category $\mathcal{C}$ is called a *monomorphism* (or an *epimorphism*) if for every two homomorphisms $g_1, g_2\colon W \to X$ (or $h_1, h_2\colon Y \to Z$), the equality $f \circ g_1 = f \circ g_2$ implies that $g_1 = g_2$ (or $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$).

Definition 7.14. An *abelian category* is a pre-abelian category such that

(1) for every monomorphism $i\colon X \to Y$, there is a homomorphism $f\colon Y \to Z$ with $\ker(f) = (X, i)$;

(2) for every epimorphism $p\colon V \to W$, there is a homomorphism $g\colon U \to V$ with $\mathrm{cok}(g) = (W, p)$.

Remark 7.15. Intuitively, one should think of an abelian category as one where it is possible to carry out all of the homological constructions that we know from categories of modules over algebras. For example, we have subobjects, quotient objects, simple objects, (short) exact sequences, the snake lemma, the Jordan-Hölder theorem, etc.

Concretely, a short exact sequence in an abelian category $\mathcal{A}$ is a sequence of homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $(A, f) = \ker(g)$ and $(C, g) = \mathrm{cok}(f)$.)

Definition 7.16. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is called *exact* if it is additive (i.e. $\mathbb{Z}$-linear) and if for every short exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$, the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

in $\mathcal{B}$ is exact.

Definition 7.17. Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category. A *fiber functor* for $\mathcal{A}$ is an exact faithful $\Bbbk$-linear functor $F\colon \mathcal{A} \to \mathbf{Vect}_{\Bbbk}$.

Definition 7.18. An abelian category $\mathcal{A}$ is called *artinian* (or *noetherian*) if every descending chain $X_0 \hookleftarrow X_1 \hookleftarrow \cdots$ (or ascending chain $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$) of subobjects of an object X of $\mathcal{A}$ stabilizes.

Lemma 7.19. *Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category with a fiber functor $F\colon \mathcal{A} \to \mathbf{Vect}_{\Bbbk}$. Then $\mathcal{A}$ is artinian and noetherian.*

Proof. For a chain $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ of subobjects of $X \in \mathrm{Ob}(\mathcal{C})$, the chain $F(X_0) \hookrightarrow F(X_1) \hookrightarrow \cdots$ stabilizes since $F(X)$ is finite-dimensional; say $F(X_i) = F(X_{i+1}) = \cdots$. As F is exact, we have

$$F(X_{j+1}/X_j) \cong F(X_{j+1})/F(X_j) = 0$$

for all $j \geq i$, whence $X_{j+1}/X_j = 0$ because F is faithful, and so $X_i = X_{i+1} = \cdots$. This proves that $\mathcal{A}$ is noetherian; the fact that $\mathcal{A}$ is artinian can be proven analogously. $\square$

8 Coend and reconstruction for abelian categories

Unless otherwise stated, in this section $\mathcal{C}$ denotes a (not necessarily monoidal) small category, $\Bbbk$ denotes a field and $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ denotes a functor.

Example 8.1. Let A be a $\Bbbk$-algebra and let $F\colon A - \mathbf{Mod} \to \mathbf{Vect}_\Bbbk^\infty$ be the functor that sends an A-module to the underlying vector space. There is an isomorphism of functors $F \cong \mathrm{Hom}_A(A, -)$, where we consider A as an A-module via left multiplication, and so Yoneda's lemma gives an isomorphism of $\Bbbk$-algebras

$$A \cong \mathrm{End}_A(A)^{\mathrm{op}} \cong \mathrm{End}\big(\mathrm{Hom}_A(A, -)\big) \cong \mathrm{End}(F).$$

In other words, the algebra A can be reconstructed from F (by taking the endomorphism algebra).

Remark 8.2. Let $\mathcal{C}$ be a small category and let $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ be a functor to the category of finite-dimensional vector spaces, for some field $\Bbbk$. Then the natural endomorphisms of F can be considered as a subspace of the product $\prod_{X\in\mathrm{Ob}(\mathcal{C})} \mathrm{End}_\Bbbk\big(F(X)\big)$, consisting of the elements $(\vartheta_X)_{X\in\mathrm{Ob}(\mathcal{C})}$ such that for every homomorphism $f\colon X \to Y$ in $\mathcal{C}$, we have $\vartheta_Y\colon F(f) = F(f) \circ \vartheta_Y$.

Another way of putting this is as follows: For a homomorphism $f\colon Y \to Z \in \mathcal{C}$, we can define a two $\Bbbk$-linear maps

$$a_f\colon \prod_X \mathrm{End}_\Bbbk\big(F(X)\big) \longrightarrow \mathrm{Hom}\big(F(Y), F(Z)\big), \qquad (e_X)_X \longmapsto F(f) \circ e_Y,$$

$$b_f\colon \prod_X \mathrm{End}_\Bbbk\big(F(X)\big) \longrightarrow \mathrm{Hom}\big(F(Y), F(Z)\big), \qquad (e_X)_X \longmapsto e_Z \circ F(f),$$

and by the universal property of the product, there are two unique k-linear maps

$$a\colon \prod_X \mathrm{End}_\Bbbk\big(F(X)\big) \longrightarrow \prod_{f\colon Y\to Z} \mathrm{Hom}_\Bbbk\big(F(Y), F(Z)\big)$$

$$b\colon \prod_X \mathrm{End}_\Bbbk\big(F(X)\big) \longrightarrow \prod_{f\colon Y\to Z} \mathrm{Hom}_\Bbbk\big(F(Y), F(Z)\big)$$

such that $\pi_f \circ a = a_f$ and $\pi_f \circ b = b_f$ for all $f\colon Y \to Z$, where π_f denotes the projection. Then $\mathrm{End}(F)$ is the equalizer of a and b, i.e. the largest subspace $i\colon E \hookrightarrow \prod_X \mathrm{End}_\Bbbk\big(F(X)\big)$ such that $a \circ i = b \circ i$.

For ease of notation, let us from now on write $E_X = \mathrm{End}_\Bbbk\big(F(X)\big)$ for all objects X of $\mathcal{C}$.

Remark 8.3. Let $\mathcal{C}$ be a small category and let $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ be a functor. We construct a coalgebra $\mathrm{End}^\vee(F)$, called the *Coend* of F, by dualizing the construction of $\mathrm{End}(F)$ in Remark 8.2. First, for a homomorphism $f\colon Y \to Z$ in $\mathcal{C}$, consider the $\Bbbk$-linear maps

$$a_f\colon \mathrm{Hom}_\Bbbk\big(F(Z), F(Y)\big) \longrightarrow \bigoplus_X E_X, \qquad g \mapsto F(f) \circ g,$$

$$b_f\colon \mathrm{Hom}_\Bbbk\big(F(Z), F(Y)\big) \longrightarrow \bigoplus_X E_X, \qquad g \mapsto g \circ F(f).$$

By the universal property of the coproduct, there are two unique $\Bbbk$-linear maps

$$a\colon \bigoplus_{f\colon Y\to Z} \mathrm{Hom}_\Bbbk\big(F(Z), F(Y)\big) \longrightarrow \bigoplus_X E_X,$$

$$b\colon \bigoplus_{f\colon Y\to Z} \mathrm{Hom}_\Bbbk\big(F(Z), F(Y)\big) \longrightarrow \bigoplus_X E_X,$$

such that $a \circ i_f = a_f$ and $b \circ i_f = b_f$ for $f\colon Y \to Z$, where i_f denotes the embedding. Then $\mathrm{End}^\vee(F)$ is the coequalizer of a and b, i.e. the largest quotient $q\colon \bigoplus_X E_X \to C$ such that $q \circ a = q \circ b$. For an object X of $\mathcal{C}$ and $\varphi \in E_X$, let us write $q_X = q \circ i_X$ and $[\varphi] = q_X(\varphi)$, where i_X denotes the embedding of E_X into $\bigoplus_X E_X$.

Lemma 8.4. *There is an isomorphism* $\mathrm{End}^\vee(F)^* \cong \mathrm{End}(F)$.

Proof. This follows from the facts that duals of direct sums are direct products (cf. Remark 7.5) and that duals of coequalizers are equalizers. $\square$

Remark 8.5. For a finite-dimensional $\Bbbk$-vector space V, there is a canonical identification

$$\mathrm{End}_\Bbbk(V) \cong V \otimes V^* \cong V^* \otimes V.$$

Using the first isomorphism, the $\Bbbk$-algebra structure on $\mathrm{End}_\Bbbk(V)$ can be defined via

$$\mu\colon V \otimes V^* \otimes V \otimes V^* \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V \otimes \mathrm{id}_V} V \otimes V^* \qquad \text{and} \qquad \eta\colon \Bbbk \xrightarrow{\mathrm{coev}_V} V \otimes V^*.$$

Similarly, we can define a $\Bbbk$-coalgebra structure on $\mathrm{End}_\Bbbk(V) \cong V^* \otimes V$ via

$$\delta\colon V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{coev}_V \otimes \mathrm{id}_V} V^* \otimes V \otimes V^* \otimes V \qquad \text{and} \qquad \eta\colon V^* \otimes V \xrightarrow{\mathrm{ev}_V} \Bbbk.$$

Observe that η can also be identified with the trace map on $\mathrm{End}_\Bbbk(V)$. This coalgebra structure is dual to the algebra structure on $\mathrm{End}_\Bbbk(V)$ under the canonical isomorphism

$$\mathrm{End}_\Bbbk(V)^* \cong (V^* \otimes V)^* \cong V^* \otimes V^{**} \cong V^* \otimes V \cong \mathrm{End}_\Bbbk(V).$$

For $X \in \mathrm{Ob}(\mathcal{C})$, we write δ_X and ε_X for the comultiplication and the counit of $E_X = \mathrm{End}_\Bbbk\big(F(X)\big)$.

Proposition 8.6. *There are linear maps*

$$\delta\colon \mathrm{End}^\vee(F) \longrightarrow \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F) \qquad \textit{and} \qquad \varepsilon\colon \mathrm{End}^\vee(F) \longrightarrow \Bbbk,$$

unique with the property that the following diagrams commute for all objects X *of* $\mathcal{C}$*:*

$$\begin{array}{ccc} E_X & \xrightarrow{\delta_X} & E_X \otimes E_X \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_X \otimes q_X} \\ \mathrm{End}^\vee(F) & \xrightarrow[\delta]{} & \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F) \end{array} \qquad\qquad \begin{array}{ccc} E_X & & \\ {\scriptstyle q_X}\downarrow & \searrow^{\varepsilon_X} & \Bbbk \\ \mathrm{End}^\vee(F) & \nearrow_{\varepsilon} & \end{array}$$

Proof. For an oject X of $\mathcal{C}$, consider the linear map

$$\overline{\delta}_X \coloneqq (q_X \otimes q_X) \circ \delta_X\colon E_X \longrightarrow \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F).$$

By the universal property of the coproduct, there is a unique linear map

$$\delta_0\colon \bigoplus\nolimits_X E_X \longrightarrow \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F)$$

such that $\delta_0 \circ i_X = \overline{\delta}_X$ for every object X of $\mathcal{C}$. We claim that $\delta_0 \circ a = \delta_0 \circ b$ (in the notation of Remark 8.3), so that there is a unique linear map

$$\delta\colon \mathrm{End}^\vee(F) \longrightarrow \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F)$$

with $\delta \circ q = \delta_0$. (Note that this implies $\delta \circ q_X = \delta \circ q \circ i_X = \delta_0 \circ i_X = \overline{\delta}_X = (q_X \otimes q_X) \circ \delta_X$, as required.)

Indeed, let $f\colon X \to Y$ be a homomorphism in $\mathcal{C}$ and fix bases $(e_i)_i$ and $(f_j)_j$ of $F(X)$ and $F(Y)$, respectively, with dual bases $(e_i^*)_i$ and $(f_j^*)_j$. Then, with $F(f)_{i,j} \coloneqq f_j^* \circ F(f)(e_i)$, we can write

$$F(f) = \sum_{i,j} F(f)_{i,j} \cdot e_i^* \otimes f_j,$$

and by the definition of $\mathrm{End}^\vee(F)$, we have for all k, ℓ the following equality, where we implicitly use the canonical isomorphism $\mathrm{Hom}_\Bbbk\big(F(Y), F(X)\big) \cong F(Y)^* \otimes F(X)$:

$$\Big[\sum\nolimits_i F(f)_{i,k} \cdot e_i^* \otimes e_\ell\Big] = [(f_k^* \otimes e_\ell) \circ F(f)] = [F(f) \circ (f_k^* \otimes e_\ell)] = \Big[\sum\nolimits_j F(f)_{\ell,j} \cdot f_k^* \otimes f_j\Big]$$

Applying this equality twice, we obtain

$$\begin{aligned}
\delta_0 \circ a_f(f_k^* \otimes e_\ell) &= \delta_Y\big(F(f) \circ (f_k^* \otimes e_\ell)\big) \\
&= \Big[\sum_{j,m} F(f)_{\ell,j} \cdot f_k^* \otimes f_m \otimes f_m^* \otimes f_j\Big] \\
&= \Big[\sum_{i,m} F(f)_{i,m} \cdot f_k^* \otimes f_m \otimes e_i^* \otimes e_\ell\Big] \\
&= \Big[\sum_{i,n} F(f)_{n,k} \cdot e_n^* \otimes e_i \otimes e_i^* \otimes e_\ell\Big] \\
&= \delta_X\big((f_k^* \otimes e_\ell) \circ F(f)\big) \\
&= \delta_0 \circ b_f(f_k^* \otimes e_\ell),
\end{aligned}$$

and therefore $\delta_0 \circ a = \delta_0 \circ b$, as required.

Similarly, the linear maps $\varepsilon_X \colon E_X \to \Bbbk$ give rise to a linear map

$$\varepsilon_0 \colon \bigoplus_Y X_Y \longrightarrow \Bbbk$$

such that $\varepsilon_0 \circ i_X = \varepsilon_X$ for all objects X of $\mathcal{C}$. Now for a homomorphism $f \colon Y \to Z$ in $\mathcal{C}$ and $g \in \mathrm{Hom}_\Bbbk\big(F(Z), F(Y)\big)$, we have

$$\varepsilon_0 \circ a_f(g) = \varepsilon_Y\big(F(f) \circ g\big) = \mathrm{tr}\,\big(F(f) \circ g\big) = \mathrm{tr}\,\big(g \circ F(f)\big) = \varepsilon_Z\big(g \circ F(f)\big) = \varepsilon_0 \circ b_f(g),$$

hence $\varepsilon_0 \circ a = \varepsilon_0 \circ b$ and there is a unique linear map

$$\varepsilon \colon \mathrm{End}^\vee(F) \longrightarrow \Bbbk$$

such that $\varepsilon \circ q = \varepsilon_0$. ☐

Proposition 8.7. *The vector space* $\mathrm{End}^\vee(F)$ *together with the linear maps*

$$\delta \colon \mathrm{End}^\vee(F) \longrightarrow \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(F) \qquad \textit{and} \qquad \varepsilon \colon \mathrm{End}^\vee(F) \longrightarrow \Bbbk$$

defined in Proposition 8.6 is a coalgebra.

Proof. Since $\mathrm{End}^\vee(F)$ is spanned by elements of the form $[\varphi]$, for $\varphi \in E_X$ and X an object of $\mathcal{C}$, the coassociativity follows if we show that $(\delta \otimes \mathrm{id}) \circ \delta$ coincides with $(\mathrm{id} \otimes \delta) \circ \delta$ when evaluated on $[\varphi]$. Using the first commutative diagram from Proposition 8.6, we obtain

$$\begin{aligned}
(\mathrm{id} \otimes \delta) \circ \delta([\varphi]) &= (\mathrm{id} \otimes \delta) \circ \delta \circ q_X(\varphi) \\
&= (\mathrm{id} \otimes \delta) \circ (q_X \otimes q_X) \circ \delta_X(\varphi) \\
&= (q_X \otimes q_X \otimes q_X) \circ (\mathrm{id} \otimes \delta_X) \circ \delta_X(\varphi) \\
&= (q_X \otimes q_X \otimes q_X) \circ (\delta_X \otimes \mathrm{id}) \circ \delta_X(\varphi) \\
&= \cdots = (\delta \otimes \mathrm{id}) \circ \delta([\varphi]),
\end{aligned}$$

as required. The counitality follows by a similar argument. ☐

Remark 8.8. For a finite-dimensional vector space V and an algebra A, it is well-known that specifying an A-module structure on V amounts to the same as giving a homomorphism of algebras $A \to \mathrm{End}_{\Bbbk}(V)$. Similarly, for a coalgebra C, a C-comodule structure on V amounts to the same as a homomorphism of coalgebras $\mathrm{End}_{\Bbbk}(V) \to C$. In particular, V is canonically an $\mathrm{End}_{\Bbbk}(V)$-comodule (via the identity coalgebra homomorphism).

Note that for $X \in \mathrm{Ob}(\mathcal{C})$, the linear map $q_X : E_X \to \mathrm{End}^\vee(F)$ is a coalgebra homomorphism by Proposition 8.6. Thus we can consider $F(X)$ as a $\mathrm{End}^\vee(F)$-comodule via the coaction map

$$F(X) \longrightarrow E_X \otimes F(X) \xrightarrow{q_X \otimes \mathrm{id}_{F(X)}} \mathrm{End}^\vee(F) \otimes F(X).$$

Hence, if we fix a basis $(e_i)_i$ of $F(X)$ then the coaction is given by $v \mapsto \sum_i [v \otimes e_i^*] \otimes e_i$. Furthermore, for a homomorphism $f : X \to Y$ in $\mathcal{C}$, the linear map $F(f) : F(X) \to F(Y)$ is a homomorphism of $\mathrm{End}^\vee(F)$-comodules. Thus $F : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ can be lifted canonically to a functor

$$\hat{F} : \mathcal{C} \longrightarrow \mathrm{End}^\vee(F) - \mathbf{comod}.$$

Writing $\mathrm{For}_{\mathrm{End}^\vee(F)} : \mathrm{End}^\vee(F) - \mathbf{comod} \to \mathbf{Vect}_{\Bbbk}$ for the forgetful functor, we have $F = \mathrm{For}_{\mathrm{End}^\vee(F)} \circ \hat{F}$.

Remark 8.9. Suppose that we are given two categories $\mathcal{C}$ and $\mathcal{D}$ with functors

$$F : \mathcal{C} \longrightarrow \mathbf{Vect}_{\Bbbk} \qquad \text{and} \qquad G : \mathcal{D} \longrightarrow \mathbf{Vect}_{\Bbbk}.$$

Further suppose that there is a functor $T : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $\gamma : F \to G \circ T$. Then for every object X of $\mathcal{C}$, we can define a coalgebra homomorphism

$$e^X_{T,\gamma} : \mathrm{End}_{\Bbbk}\big(F(X)\big) \longrightarrow \mathrm{End}_{\Bbbk}\big(G \circ T(X)\big), \qquad \varphi \mapsto \gamma_Y \circ \varphi \circ \gamma_X^{-1}.$$

Furthermore, we can define a linear map $e_{T,\gamma} : \mathrm{End}^\vee(F) \to \mathrm{End}^\vee(G)$ with

$$e_{T,\gamma}([\varphi]) = [\gamma_X \circ \varphi \circ \gamma_X^{-1}],$$

that is $e_{T,\gamma} \circ q_X = q_{T(X)} \circ e_{T,\gamma^X}$. Indeed, we have

$$[\gamma_X \circ F(f) \circ g \circ \gamma_X^{-1}] = [F(f) \circ \gamma_Y \circ g \circ \gamma_X^{-1}] = [\gamma_Y \circ g \circ \gamma_X^{-1} \circ F(f)] = [\gamma_Y \circ g \circ F(f) \circ \gamma_Y^{-1}]$$

for a homomorphism $f : X \to Y$ in $\mathcal{C}$ and $g \in \mathrm{Hom}_{\Bbbk}\big(F(Y), F(X)\big)$, whence the universal property of the equalizer yields the well-definedness of $e_{T,\gamma}$. Since $e_{T,\gamma}$ is a coalgebra homomorphism for all objects X of $\mathcal{C}$, it is straightforward to see (using the commutative diagrams from Proposition 8.6) that $e_{T,\gamma}$ is a coalgebra homomorphism.

Now let us write $\mathbf{Cat}/\mathbf{Vect}_{\Bbbk}$ for the category of *small categories over* $\mathbf{Vect}_{\Bbbk}$, that is, the category whose objects are pairs $(\mathcal{C}, F)$ of a small category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$. The homomorphisms are given by pairs (T, γ) as above. Then we can define a functor

$$\mathrm{End}^\vee : \mathbf{Cat}/\mathbf{Vect}_{\Bbbk} \longrightarrow \mathbf{Coalg}_{\Bbbk}$$

from $\mathbf{Cat}/\mathbf{Vect}_{\Bbbk}$ to the category of coalgebras over $\Bbbk$ by sending an object $(\mathcal{C}, F)$ to the coalgebra $\mathrm{End}^\vee(F)$ and a homomorphism (T, γ) to the coalgebra homomorphism $e_{T,\gamma}$

Theorem 8.10. *Let $\mathcal{C}$ be a $\Bbbk$-linear abelian category and $F : \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ a fiber functor. Then*

$$\hat{F} : \mathcal{C} \longrightarrow \mathrm{End}^\vee(F) - \mathbf{comod}$$

is an equivalence of categories.

Proof (sketch). Let us start with a general observation: For any coablgebra (C, δ, ε) and a C-comodule M, we can consider $C \otimes M$ as a C-comodule with coaction map $\delta \otimes \mathrm{id}_M : C \otimes M \to C \otimes C \otimes M$, and the coaction map $\delta_M : M \to C \otimes M$ thus becomes a homomorphism of C-comodules. Similarly, we can consider $C \otimes C \otimes M$ as a C-comodule with action map $\delta \otimes \mathrm{id}_C \otimes \mathrm{id}_M$, making both

$$\mathrm{id}_C \otimes \delta_M : C \otimes M \to C \otimes C \otimes M \qquad \text{and} \qquad \delta \otimes \mathrm{id}_M : C \otimes M \to C \otimes C \otimes M$$

homomorphisms of C-comodules. Furthermore, we claim that (M, δ_M) is the kernel of

$$\vartheta_M \coloneqq \mathrm{id}_C \otimes \delta_M - \delta \otimes \mathrm{id}_M.$$

Indeed, we have $(\mathrm{id}_C \otimes \delta_M)\delta_M = (\delta \otimes \mathrm{id}_M) \circ \delta_M$ by coassociativity, whence $\vartheta_M \circ \delta_M = 0$, and for a comodule M' with a homomorphism $\delta' : M' \to C \otimes M$ such that $\vartheta_M \otimes \delta' = 0$, we define $u \coloneqq (\varepsilon \otimes \mathrm{id}_M) \circ \delta : M' \to M$ and verify that

$$\begin{aligned}\delta_M \circ u = \delta_M \circ (\varepsilon \otimes \mathrm{id}_M) \circ \delta' &= (\varepsilon \otimes \mathrm{id}_C \otimes \mathrm{id}_M) \circ (\mathrm{id}_C \otimes \delta_M) \circ \delta' \\ &= (\varepsilon \otimes \mathrm{id}_C \otimes \mathrm{id}_M) \circ (\delta \otimes \mathrm{id}_M) \circ \delta' = (\mathrm{id}_C \otimes \mathrm{id}_M) \circ \delta' = \delta',\end{aligned}$$

as required.

Now set $C = \mathrm{End}^\vee(F)$. Our goal is to define (functorially in M) a homomorphism $\hat{\vartheta}_M : X_M \to Y_M$ in a suitably enlarged version of $\mathcal{C}$ (called the *ind-completion*) such that $\hat{F}(\hat{\vartheta}_M) = \vartheta_M$. Then we can define a quasi-inverse of $\hat{F}$ as the functor $M \mapsto \ker(\hat{\vartheta}_M)$.

In order to do this, first note that since $\mathcal{C}$ is $\Bbbk$-linear, we can define for every object X of $\mathcal{C}$ and every $\Bbbk$-vector space V an object $X \otimes V$ of $\mathcal{C}$ (as the coproduct of $\dim V$ copies of X) such that

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes V, -) \cong \mathrm{Hom}_{\Bbbk}\big(V, \mathrm{Hom}_{\mathcal{C}}(X, -)\big)$$

as functors from $\mathcal{C}$ to **Vect**, and via Yoenda's lemma, we can define for every homomorphism $f : X \to Y$ and every linear map $g : V \to W$ a homomorphism $f \otimes g : X \otimes V \to Y \otimes W$. Furthermore, since F is $\Bbbk$-linear, we have $F(X \otimes V) \cong F(X) \otimes V$ and $F(f \otimes g) = F(f) \otimes g$. (In other words, $\mathcal{C}$ is a module category over $\mathbf{Vect}_\Bbbk$ and F is a module functor.)

Now consider the ind-object $\hat{E} \coloneqq \bigoplus_x X \otimes F(X)^*$ of $\mathcal{C}$, and note that $\hat{E}$ is sent by F to the vector space $\bigoplus_X F(X) \otimes F(X)^* \cong \bigoplus_X E_X$. Let $\hat{C}$ be the coequalizer of the homomorphisms

$$\hat{a}_f : X \otimes F(Y)^* \xrightarrow{f \otimes \mathrm{id}_{F(Y)^*}} Y \otimes F(Y)^* \subseteq \hat{E} \qquad \text{and} \qquad \hat{b}_f : X \otimes F(Y)^* \xrightarrow{\mathrm{id}_X \otimes F(f)^*} X \otimes F(X)^* \subseteq \hat{E},$$

where f runs through all homomorphisms $f : X \to Y$ in $\mathcal{C}$. Then, by construction, $\hat{F}(\hat{C}) \cong C$ as a C-comodule. Furthermore, the coevaluation maps

$$X \otimes F(X)^* \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_{F(X)} \otimes \mathrm{id}_{F(X)^*}} X \otimes F(X)^* \otimes F(X) \otimes F(X)^*$$

induce a homomorphism $\hat{\delta} : \hat{C} \to \hat{C} \otimes C$ such that $F(\hat{\delta}) = \delta : C \to C \otimes C$ identifies with the comultiplication map.

For every C-comodule M, we can now set

$$\hat{\vartheta}_M = \mathrm{id}_{\hat{C}} \otimes \delta_M - \hat{\delta} \otimes \mathrm{id}_M : \quad \hat{C} \otimes M \longrightarrow \hat{C} \otimes C \otimes M,$$

so that $F(\hat{\vartheta}_M) = \vartheta_M$, and define a functor from $C-\textbf{comod}$ to $\mathcal{C}$ via $M \mapsto \ker(\hat{\vartheta}_M)$. (Note that this is functorial by the universal property of the kernel, and because $\hat{\vartheta}_M$ is natural in M.) This defines a quasi-inverse for $\hat{F} : \mathcal{C} \longrightarrow C - \textbf{comod}$ because

$$\hat{F}\big(\ker(\hat{\vartheta}_M)\big) = \ker \hat{F}(\hat{\vartheta}_M) = \ker(\vartheta_M) \cong M$$

since $\hat{F}$ is exact. □

Remark 8.11. Let C be a $\mathbb{k}$-coalgebra and write $F = \mathrm{For}_C\colon C - \mathbf{comod} \longrightarrow \mathbf{Vect}_{\mathbb{k}}$ for the forgetful functor. For every C-comodule X, we have a coalgebra homomorphism $\varphi_X\colon E_X = \mathrm{End}_{\mathbb{k}}(X) \to C$ corresponding to the coaction map $X \to C \otimes X$, and by the universal property of the coproduct, we obtain a homomorphism of coalgebras $\varphi\colon \bigoplus_X E_X \longrightarrow C$ such that $\varphi_X = \varphi \circ i_X$ for every C-comodule X. If is straigntforward to check that $\varphi \circ a = \varphi \circ b$, with notation as in Remark 8.3, and so φ induces a coalgebra homomorphism $u\colon \mathrm{End}^\vee(F) \to C$. It turns out that u is an isomorphism, so $C \cong \mathrm{End}^\vee(F)$.

Corollary 8.12. *The functor*

$$\mathrm{End}^\vee\colon \mathbf{Cat}/\mathbf{Vect}_{\mathbb{k}} \longrightarrow \mathbf{Coalg}_{\mathbb{k}}$$

induces an equivalence between $\mathbf{Coalg}_{\mathbb{k}}$ *and the category of* $\mathbb{k}$*-linear abelian categories with a fiber functor. In particular, there is a bijection between coalgebras, up to isomorphism, and* $\mathbb{k}$*-linear abelian categories* $\mathcal{C}$ *with a fiber functor* F*, up to equivalence, given by*

$$(\mathcal{C}, F) \mapsto \mathrm{End}^\vee(F) \qquad \text{and} \qquad C \mapsto (C - \mathbf{comod}, \mathrm{For}_C).$$

Proof. This follows from Theorem 8.10 and Remark 8.11 ☐

9 Reconstruction for monoidal categories

Remark 9.1. Recall that the category **Cat** of small categories has a monoidal structure via the product of categories from Remark 1.5, and that a monoid in **Cat** is the same as a strict monoidal category (see Example 6.2). For two small categories $\mathcal{C}, \mathcal{D}$ and functors $F\colon \mathcal{C} \to \mathbf{Vect}_{\mathbb{k}}$, $G\colon \mathcal{D} \to \mathbf{Vect}_{\mathbb{k}}$, we define the product functor

$$F \times G\colon \mathcal{C} \otimes \mathcal{D} \longrightarrow \mathbf{Vect}_{\mathbb{k}}$$

by $(F \times G)(X, Y) = F(X) \otimes G(Y)$ and $(f, g) \mapsto F(f) \otimes F(g)$, for objects X, Y and homomorphisms f, g in $\mathcal{C}$ and $\mathcal{D}$, respectively. This endows the category $\mathbf{Cat}/\mathbf{Vect}_{\mathbb{k}}$ from Remark 8.9 with a monoidal structure, and a monoid in $\mathbf{Cat}/\mathbf{Vect}_{\mathbb{k}}$ is the same as a strict monoidal category $\mathcal{C}$ with a monoidal functor $F\colon \mathcal{C} \to \mathbf{Vect}_{\mathbb{k}}$.

Theorem 9.2. *For categories* $\mathcal{C}$ *and* $\mathcal{D}$ *with functors* $F\colon \mathcal{C} \to \mathbf{Vect}_{\mathbb{k}}$ *and* $G\colon \mathcal{D} \longrightarrow \mathbf{Vect}_{\mathbb{k}}$*, there is a coalgebra isomorphism*

$$\mathrm{End}^\vee(F \times G) \cong \mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(G).$$

More precisely, the functor $\mathrm{End}^\vee\colon \mathbf{Cat}/\mathbf{Vect}_{\mathbb{k}} \longrightarrow \mathbf{Coalg}_{\mathbb{k}}$ *is monoidal.*

Proof (sketch). For all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, there is a coalgebra isomorphism

$$\mathrm{End}_{\mathbb{k}}\big(F(X)\big) \otimes \mathrm{End}_{\mathbb{k}}\big(G(Y)\big) \longrightarrow \mathrm{End}_{\mathbb{k}}\big(F(X) \otimes G(Y)\big) = \mathrm{End}_{\mathbb{k}}\big((F \times G)(X, Y)\big), \qquad \varphi \otimes \psi \mapsto \varphi \otimes \psi.$$

These isomorphisms define mutually inverse linear maps

$$\mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(G) \longleftrightarrow \mathrm{End}^\vee(F \times G).$$

More specifically, the map $\mathrm{End}^\vee(F) \otimes \mathrm{End}^\vee(G) \to \mathrm{End}^\vee(F \times G)$ is given by $[\varphi] \otimes [\psi] \mapsto [\varphi \otimes \psi]$. It is straightforward to see that this map is a coalgebra homomorphism. ☐

Corollary 9.3. *If* $\mathcal{C}$ *is monoidal and* $F\colon \mathcal{C} \to \mathbf{Vect}_{\mathbb{k}}$ *can be endowed with the structure of a monoidal functor then* $\mathrm{End}^\vee(F)$ *has a canonical bialgebra structure.*

Proof. By the Strictness Theorem 3.3 and the functoriality of $\mathrm{End}^\vee$ (see Remark 8.9), we may assume that $\mathcal{C}$ is strict. Then $(\mathcal{C}, F)$ is a monoid in $\mathbf{Cat}/\mathbf{Vect}_{\mathbb{k}}$ by Remark 9.1. Thus, by Lemma 6.3 and Theorem 9.2, $\mathrm{End}^\vee(F)$ is a monoid in $\mathbf{Coalg}_{\mathbb{k}}$, i.e. a bialgebra. ☐

Remark 9.4. For $\mathcal{C}$ monoidal, $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ a monoidal functor and X, Y objects of $\mathcal{C}$, there are canonical isomorphisms

$$\mathrm{End}_{\Bbbk}\big(F(X)\big) \otimes \mathrm{End}_{\Bbbk}\big(F(Y)\big) \cong \mathrm{End}_{\Bbbk}\big(F(X) \otimes F(Y)\big) \cong \mathrm{End}_{\Bbbk}\big(F(X \otimes Y)\big)$$

such that

$$\vartheta \otimes \psi \mapsto \varphi_{X,Y}^{-1} \circ (\vartheta \otimes \psi) \circ \varphi_{X,Y} =: \vartheta \underline{\otimes} \psi.$$

The multiplication on $\mathrm{End}^{\vee}(F)$ is given by $[\vartheta] \otimes [\psi] \mapsto [\vartheta \underline{\otimes} \psi]$.

Lemma 9.5. *Let $\mathcal{C}$ be monoidal and $(F, \varphi, \varepsilon)\colon \mathcal{C} \to \mathbf{Vect}_{\Bbbk}$ a monoidal functor. Then the linear maps*

$$\varphi_{X,Y}\colon F(X \otimes Y) \longrightarrow F(X) \otimes F(Y) \qquad \text{and} \qquad \varepsilon\colon F(\mathbf{1}) \longrightarrow \Bbbk$$

are isomorphisms of $\mathrm{End}^{\vee}(F)$-comodules, for all objects X, Y of $\mathcal{C}$. In particular,

$$(\hat{F}, \varphi, \varepsilon)\colon \mathcal{C} \longrightarrow \mathrm{End}^{\vee}(F) - \mathbf{comod}$$

is a monoidal functor.

Proof. Let us fix bases $(e_i)_i$ of $F(X)$ and $(f_j)_j$ of $F(Y)$. Then we can define a basis $(b_{ij})_{i,j}$ of $F(X \otimes Y)$ via $b_{ij} = \varphi_{X,Y}^{-1}(e_i \otimes f_j)$. Writing $C = \mathrm{End}^{\vee}(F)$ and $\delta_{F(X\otimes Y)}\colon F(X \otimes Y) \to C \otimes F(X \otimes Y)$ for the coaction map, we have

$$(\mathrm{id}_C \otimes \varphi_{X,Y}) \circ \delta_{F(X\otimes Y)}(b_{i,j}) = (\mathrm{id}_C \otimes \varphi_{X,Y})\Big(\sum\nolimits_{k,\ell} [b_{i,j} \otimes b_{k,\ell}^*] \otimes b_{k,\ell}\Big) = \sum\nolimits_{k,\ell} [b_{i,j} \otimes b_{k,\ell}^*] \otimes e_k \otimes f_\ell.$$

The coaction map on $F(X) \otimes F(Y)$ is given by

$$\delta_{F(X)\otimes F(Y)} = (\mu \otimes \mathrm{id}_{F(X)\otimes F(Y)}) \circ s \circ (\delta_{F(X)} \otimes \delta_{F(Y)})$$

according to Remark 6.6, where we write $s = (\mathrm{id}_C \otimes \beta_{F(X),C} \otimes \mathrm{id}_{F(Y)})$, and so we have

$$\begin{aligned}
\delta_{F(X)\otimes F(Y)} \circ \varphi(b_{i,j}) &= (\mu \otimes \mathrm{id}_{F(X)\otimes F(Y)}) \circ s\Big(\sum\nolimits_{k,\ell} [e_i \otimes e_k^*] \otimes e_k \otimes [f_j \otimes f_\ell^*] \otimes f_\ell\Big) \\
&= \sum\nolimits_{k,\ell} \big[(e_i \otimes e_k^*) \underline{\otimes} (f_j \otimes f_\ell^*)\big] \otimes e_k \otimes f_\ell \\
&= \sum\nolimits_{k,\ell} \big[b_{i,j} \otimes b_{k,\ell}^*\big] \otimes e_k \otimes f_\ell \\
&= (\mathrm{id}_C \otimes \varphi_{X,Y}) \circ \delta_{F(X\otimes Y)}(b_{i,j})
\end{aligned}$$

because

$$(e_i \otimes e_k^*) \underline{\otimes} (f_j \otimes f_\ell^*) = \varphi_{X,Y}^{-1} \circ \big((e_i \otimes f_j) \otimes (e_k \otimes f_\ell)^*\big) \circ \varphi_{X,Y} = b_{i,j} \otimes b_{k,\ell}^*.$$

This implies that $\varphi_{X,Y}$ is a homomorphism of $\mathrm{End}^{\vee}(F)$-comodules. Similarly, for $x \in F(\mathbf{1})$ such that $\varepsilon(x) = 1 \in \Bbbk$, we have

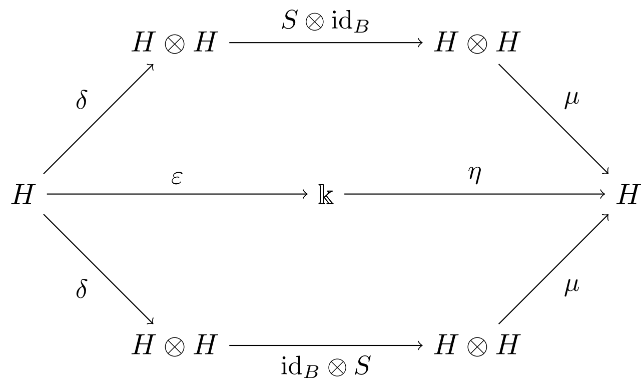
$$(\mathrm{id}_C \otimes \varepsilon) \circ \delta_{F(\mathbf{1})}(x) = [x \otimes x^*] \otimes 1 = [\mathrm{id}_{F(\mathbf{1})}] \otimes 1 = (\eta \otimes \mathrm{id}_{\Bbbk})(x),$$

whence $\varepsilon\colon F(X) \to \Bbbk$ is a homomorphism of $\mathrm{End}^{\vee}(F)$-comodules. □

Definition 9.6. A *tensor category* over $\Bbbk$ is a $\Bbbk$-linear abelian $\Bbbk$-linearly monoidal category.

Corollary 9.7. *There is a bijection between tensor categories over $\Bbbk$ with a monoidal fiber functor, up to equivalence, and bialgebras over $\Bbbk$, up to isomorphism.*

Definition 9.8. An *antipode* on a bialgebra algebra B is a $\Bbbk$-linear map $S\colon B \to B$ such that the following diagram commutes:



A pair (B, S) of a bialgebra and an antipode is called a *Hopf algebra*.

For time reasons, we do not prove the following lemma and instead refer the reader to Proposition 9.3.3 and the discussion after Proposition 9.23 in [Maj95].

Lemma 9.9. *Let H be a Hopf algebra with antipode $S\colon H \to H$.*

(1) For every finite-dimensional H-module M, we can define an H-module structure on M^* via

$$a \cdot \xi(x) = \xi\big(S(x) \cdot v\big).$$

(2) For every finite-dimensional H-comodule N, we can define an H-comodule structure on N^* with coaction map $\delta_{N^*}\colon N^* \to H \otimes N^*$ given by

$$\delta_{N^*}(\xi) = (S \otimes \xi \otimes \mathrm{id}_{N^*}) \circ (\delta_N \otimes \mathrm{id}_{N^*}) \circ \mathrm{coev}_N(1).$$

*For each of the constructions in (1) and (2), the standard evaluation and coevaluation maps for M and N are homomorphisms of H-modules or H-comodules, respectively. In particular, the monoidal categories $H -$ **mod** and $H -$ **comod** have left-duals.*

Remark 9.10. Let H be a Hopf algebra with antipode $S\colon H \to H$, let M be an H-module and let N be an H-comodule. If S is invertible then we can define an alternative H-module strucure on the dual space ${}^*M = M^*$ via $(a \cdot \xi)(x) = \xi(S^{-1} \cdot x)$, and this makes ${}^*M$ a right dual of M in $H -$ **mod**. Similarly, we can define a right dual comodule ${}^*N$ of N. Thus, if the antipode S is invertible then the monoidal categories $H -$ **mod** and $H -$ **comod** are rigid.

Lemma 9.11. *Suppose that every object of $\mathcal{C}$ admits a left dual. Then there is a unique linear map $S\colon \mathrm{End}^\vee(F) \longrightarrow \mathrm{End}^\vee(F)$ with $S([\varphi]) = [\varphi^*]$ for an object X of $\mathcal{C}$ and $\varphi \in E_X$, where we view φ^* as an element of E_{X^*}. Furthermore, S is an antipode on $\mathrm{End}^\vee(F)$.*

Proof. The map S is well-defined because we have

$$\Big[\big(F(f) \circ g\big)^*\Big] = \big[g^* \circ F(f)^*\big] = \big[g^* \circ F(f^*)\big] = \big[F(f^*) \circ g\big] = \big[F(f)^* \circ g\big] = \Big[\big(g \circ F(f)\big)^*\Big]$$

for all homomorphisms $f\colon X \to Y$ in $\mathcal{C}$ and all $g \in \mathrm{Hom}_{\Bbbk}\big(F(Y), F(X)\big)$. For an object X of $\mathcal{C}$ and a basis $(e_i)_i$ of $F(X)$, we have

$$\begin{aligned} \mu \circ (\mathrm{id}_C \otimes S) \circ \delta(e_i^* \otimes e_j) &= \mu \circ (\mathrm{id}_C \otimes S)\Big(\sum\nolimits_k [e_i^* \otimes e_k] \otimes [e_k^* \otimes e_j]\Big) \\ &= \mu\Big(\sum\nolimits_k [e_i^* \otimes e_k] \otimes [e_j \otimes e_k^*]\Big) \\ &= \sum\nolimits_k \big[\varphi_{X,X^*}^{-1} \circ \big((e_i^* \otimes e_j) \otimes (e_k \otimes e_k^*)\big) \circ \varphi_{X,X^*}\big], \end{aligned}$$

where $e_j \otimes e_k^*$ is the endomorphism of $F(X)^*$ with $(e_j \otimes e_k^*)(e_m^*) = \delta_{j,m} \cdot e_k^*$, and $(e_i^* \otimes e_j) \otimes (e_k \otimes e_k^*)$ is the endomorphism of $F(X) \otimes F(X)^*$ given by $\big((e_i^* \otimes e_j) \otimes (e_k \otimes e_k^*)\big)(e_m \otimes e_n^*) = \delta_{im}\delta_{jn} \cdot (e_k \otimes e_k^*)$. Now we have

$$\sum_k \varphi_{X,X^*}^{-1} \circ (e_k \otimes e_k^*) = \varphi_{X,X^*}^{-1} \circ \mathrm{coev}_{F(X)} = F(\mathrm{coev}_X)$$

as homomorphisms from $\Bbbk$ to $F(X \otimes X^*)$, and we can similarly view $e_i^* \otimes e_j$ as a homomorphism from $F(X) \otimes F(X)^*$ to $\Bbbk$. Using these identifications, we further obtain

$$\begin{aligned}
\mu \circ (\mathrm{id} \otimes S) \circ \delta(e_i^* \otimes e_j) &= \textstyle\sum_k \big[\varphi_{X,X^*}^{-1} \circ \big((e_i^* \otimes e_j) \otimes (e_k \otimes e_k^*)\big) \circ \varphi_{X,X^*}\big] \\
&= \textstyle\sum_k \big[\varphi_{X,X^*}^{-1} \circ (e_k \otimes e_k^*) \circ (e_i^* \otimes e_j) \circ \varphi_{X,X^*}\big] \\
&= \big[F(\mathrm{coev}_X) \circ (e_i^* \otimes e_j) \circ \varphi_{X,X^*}\big] \\
&= \big[(e_i^* \otimes e_j) \circ \varphi_{X,X^*} \circ F(\mathrm{coev}_X)\big] \\
&= \textstyle\sum_k \big[(e_i^* \otimes e_j) \circ (e_k \otimes e_k^*)\big] \\
&= \delta_{ij} \cdot [\mathrm{id}_{F(\mathbf{1})}].
\end{aligned}$$

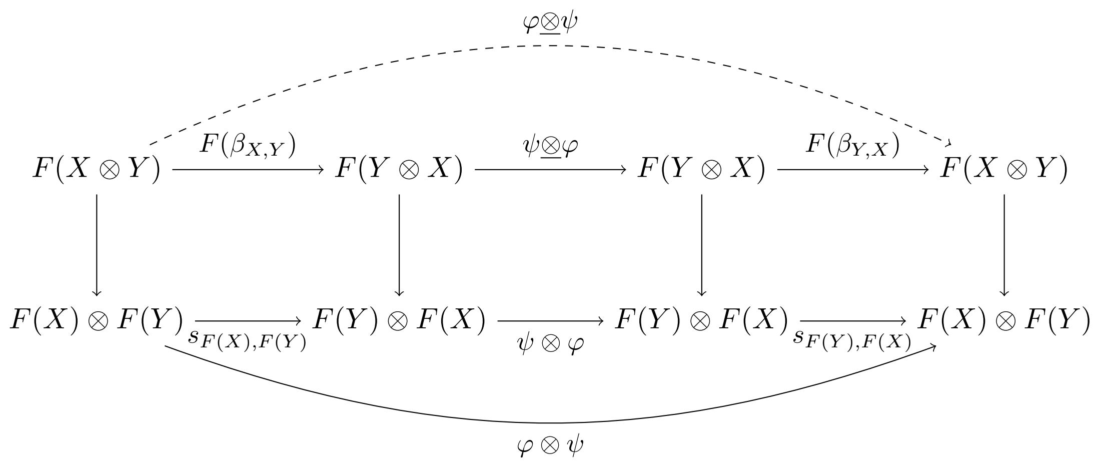
We also have $\eta \circ \varepsilon(e_i^* \otimes e_j) = \delta(\delta_{ij}) = \delta_{ij} \cdot [\mathrm{id}_{F(\mathbf{1})}]$ and so $\eta \circ \varepsilon = \mu \circ (\mathrm{id}_C \otimes S) \circ \delta$, as required. The equality $\eta \circ \varepsilon = \mu \circ (S \otimes \mathrm{id}_C) \circ \delta$ is proven analogously. □

Corollary 9.12. *(1) There is a bijection between Hopf algebras over $\Bbbk$, up to isomorphism, and tensor categories over $\Bbbk$ with left duals and a monoidal fiber functor, up to equivalence.*

(2) There is a bijection between Hopf algebras with invertible antipode, up to isomorphism, and rigid tensor categories over $\Bbbk$ with a monoidal fiber functor, up to equivalence.

Lemma 9.13. *If $\mathcal{C}$ admits a symmetric braiding β and $(F, \gamma, \varepsilon)\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$ is a symmetric monoidal fiber functor then the bialgebra $\mathrm{End}^\vee(F)$ is commutative.*

Proof. For objects X, Y of $\mathcal{C}$ and linear endomorphisms φ and ψ of $F(X)$ and $F(Y)$, respectively, consider the following commutative diagram, where all vertical arrows are induced by γ and we write s for the canonical symmetric braiding on $\mathbf{Vect}_\Bbbk$:



Here, the middle square commutes by the definition of $\psi \underline{\otimes} \varphi$ and the left and right hand side squares commute because F is symmetric. The composition along the bottom row is equal to $\varphi \otimes \psi$ by the definition of s, and so the composition of the top row is equal to $\varphi \underline{\otimes} \psi$, again by definition. This implies that in $\mathrm{End}^\vee(F)$, we have

$$[\varphi] \cdot [\psi] = [\varphi \underline{\otimes} \psi] = [F(\beta_{X,Y}) \circ (\psi \underline{\otimes} \varphi) \circ F(\beta_{Y,X})] = [F(\beta_{Y,X}) \circ F(\beta_{X,Y}) \circ (\psi \underline{\otimes} \varphi)] = [\psi \underline{\otimes} \varphi] = [\psi] \cdot [\varphi]$$

because β is symmetric; hence $\mathrm{End}^\vee(F)$ is commutative. □

Corollary 9.14. *There is a bijection between symmetric tensor categories over $\Bbbk$ with a symmetric monoidal fiber functor, up to equivalence, and commutative bialgebras over $\Bbbk$, up to isomorphism.*

Remark 9.15. For every commutative Hopf algebra, the antipode is an invertible algebra homomorphism. This corresponds to the statement that in a symmetric monoidal category with left duals, right duals must also exist, and taking left-duals is a monoidal functor.

Corollary 9.16. *There is a bijection between rigid symmetric tensor categories over $\Bbbk$ with a symmetric monoidal fiber functor, up to equivalence, and commutative Hopf algebras over $\Bbbk$, up to isomorphism.*

Remark 9.17. Let $(H, \mu_H, \varepsilon_H, \delta_H, \eta_H, S)$ be a commutative Hopf $\Bbbk$-algebra. For any commutative $\Bbbk$-algebra (A, μ_A, η_A), we can define a group structure on

$$\mathbf{G}(A) := \mathrm{Hom}_{\Bbbk-\mathbf{Alg}}(H, A)$$

with neutral element $1_{\mathbf{G}(A)} = \eta_A \circ \varepsilon_H$ and multiplication given by

$$f \cdot g := \mu_A \circ (f \otimes g) \circ \delta_H.$$

The inverse of $f \in \mathbf{G}(A)$ is $f^{-1} = f \circ S$. Writing $\mathbf{CommAlg}_{\Bbbk}$ for the category of commutative $\Bbbk$-algebras, this construction gives rise to a functor

$$\mathbf{G} = \mathbf{G}_H : \mathbf{CommAlg}_{\Bbbk} \longrightarrow \mathbf{Grp}$$

which we call the $\Bbbk$-*group scheme* corresponding to H.

For a finite-dimensional $\Bbbk$-vector space M, let us also consider the functor

$$\hat{M} : \mathbf{CommAlg}_{\Bbbk} \longrightarrow \mathbf{Set}, \qquad M \mapsto A \otimes M.$$

A $\mathbf{G}$-*module* structure on M is a natural transformation $\mathbf{G} \times \hat{M} \longrightarrow \hat{N}$ such that the components $\mathbf{G}(A) \times (A \otimes M) \to A \otimes M$ afford A-linear $\mathbf{G}(A)$-module structures on $M \otimes A$ for all commutative $\Bbbk$-algebras A. For any H-comodule (M, δ_M), we can define a $\mathbf{G}$-module structure on M via

$$f \cdot (m \otimes 1) = (f \otimes \mathrm{id}_M) \circ \delta_M$$

for $f \in \mathbf{G}(A)$ and $m \in M$, extended by A-linearity. This gives rise to a monoidal equivalence between $H - \mathbf{comod}$ and the category $\mathbf{Rep}(\mathbf{G})$ of $\mathbf{G}$-modules. In view of the above results, there is a bijection between $\Bbbk$-group schemes, up to isomorphism, and $\Bbbk$-linear abelian rigid symmetric monoidal categories. Two important special cases are as follows:

(1) If G is a finite group then $\Bbbk[G]$ is a Hopf algebra with comultiplication, counit and antipode given by

$$g \mapsto g \otimes g, \qquad g \mapsto 1, \qquad g \mapsto g^{-1}$$

for $g \in G$, respectively, extended by $\Bbbk$-linearity. The dual Hopf algebra $\Bbbk[G]^*$ is commutative, and it is straightforward to check that the corresponding group scheme $\mathbf{G}$ satisfies $\mathbf{G}(\Bbbk) = G$. Thus, we have a monoidal equivalence between $\mathbf{Rep}_{\Bbbk}(G)$ and $\mathbf{Rep}(\mathbf{G})$ (since both categories are equivalent to $\Bbbk[G] - \mathbf{mod} \cong \Bbbk[G]^* - \mathbf{comod}$).

(2) If $\Bbbk$ is algebraically closed and H is reduced and finitely-generated then $G(\Bbbk) = \mathrm{Spec}(H)$ is an affine algebraic group over $\Bbbk$. The category $\mathbf{Rep}(\mathbf{G})$ is equivalent to the category of rational representations of $\mathbf{G}(\Bbbk)$.

For a general $\Bbbk$-linear abelian rigid symmetric monoidal category with a symmetric monoidal fiber functor F, one can check that the group

$$\mathbf{G}(\Bbbk) = \mathrm{Hom}_{\Bbbk-\mathbf{Alg}}\big(\mathrm{End}^\vee(F), \Bbbk\big) \subseteq \mathrm{End}^\vee(F)^* \cong \mathrm{End}(F)$$

identifies with the set of monoidal natural endomorphisms of F, which is indeed a group by rigidity (cf. Lemma 4.23).

In view of the above results, a rigid symmetric tensor category $\mathcal{C}$ over $\Bbbk$ is equivalent to the category of comodules over a commutative Hopf algebra (or equivalently, to the category of representations of a $\Bbbk$-group scheme) if and only if it admits a symmetric monoidal fiber functor $F\colon \mathcal{C} \to \mathbf{Vect}_\Bbbk$. Thus, it becomes important to construct fiber functors for monoidal categories. An important result along these lines is the following theorem of P. Deligne:

Theorem 9.18. *Let $\Bbbk$ be a field of characteristic 0. For a $\Bbbk$-linear rigid symmetric monoidal category $\mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}) \cong \Bbbk$, the following are equivalent:*

(1) *$\mathcal{C}$ admits a $\Bbbk$-linear exact faithful symmetric monoidal functor $F\colon \mathcal{C} \longrightarrow A-\mathbf{mod}$ for some commutative $\Bbbk$-algebra A;*

(2) *every object of $\mathcal{C}$ has non-negative integer dimension;*

(3) *every non-zero object of $\mathcal{C}$ has positive integer dimension;*

(4) *for every object X of $\mathcal{C}$, there is $n > 0$ such that the n-th exterior power $\Lambda^n X$ of X vanishes.*

In points (2) and (3), the dimension of an object of X of $\mathcal{C}$ is defined as

$$\dim X = \mathrm{ev}_X \circ \beta_{X,X^*} \circ \mathrm{coev}_X \in \mathrm{End}\mathcal{C}(\mathbf{1}) \cong \Bbbk.$$

Since $\mathcal{C}$ is symmetric, there is for every object X of $\mathcal{C}$ an algebra homomorphism

$$\varphi\colon \Bbbk[S_n] \longrightarrow \mathrm{End}_{\mathcal{C}}(X^{\otimes n}),$$

and the exterior power $\Lambda^n X$ is defined as the image of the endomorphism $\varphi\big(\sum_\sigma \mathrm{sign}(\sigma)\cdot\sigma\big)$. Under the equivalent conditions (1)–(4), reconstruction theory implies that $\mathcal{C}$ is equivalent to a category of representations of a group scheme over A.

If we take $\Bbbk = \mathbb{C}$ then we can also characterize a larger class of tensor categories via a simple growth assumption: We say that an object X of a tensor category $\mathcal{C}$ has sub-exponential growth if there exists $x > 0$ such that the composition length of $X^{\otimes n}$ is bounded by x^n for $n > 0$.

Theorem 9.19 (Deligne)**.** *For a rigid symmetric monoidal category $\mathcal{C}$ over $\mathbb{C}$, the following are equivalent:*

(1) *$\mathcal{C}$ admits an exact faithful $\mathbb{C}$-linear symmetric monoidal functor $F\colon \mathcal{C} \to \mathbf{SVect}_{\mathbb{C}}$ to the category of super vector spaces;*

(2) *for every object X of $\mathcal{C}$, there is a partition λ such that the Schur functor $S^\lambda X$ vanishes;*

(3) *every object of $\mathcal{C}$ has sub-exponential growth.*

Potential topics for talks

The topics that appear in gray have already been chosen.

(1) pivotal categories, traces and dimension;

(2) Frobenius-Perron dimension;

(3) monoidal categories by generators and relations (via diagrams);

(4) the Temperley-Lieb category (and knot invariants);

(5) tensor triangular geometry;

(6) the geometric Satake equivalence;

(7) negligible morphisms and semisimplification;

(8) the Drinfel'd double and the Drinfel'd center;

(9) $\mathfrak{sl}_n$-webs;

(10) interpolation categories;

(11) $\cdots$

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