National University of Singapore — Jonathan Gruber
Semester 2, academic year 2022 / 2023 — March 24, 2023

Exercise 1. – *to be handed in by 17 February 2023*

Give a detailed proof of the strictness theorem (Theorem 3.3 in the lecture notes).

In more detail, prove that any monoidal category $(\mathcal{C}, \mathbf{1}, \otimes, \alpha, \lambda, \rho)$ is monoidally equivalent to the strict monoidal category $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C})$ of right $\mathcal{C}$-module endofuctors of $\mathcal{C}$ (where we consider $\mathcal{C}$ as a right module category over $\mathcal{C}$, as in Example 2.3(1) in the lecture notes).

Remark: In view of Remark 1.15(3), it suffices to construct a monoidal functor

$$(F, \varphi, \varepsilon) \colon \mathcal{C} \longrightarrow \mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C})$$

and a functor $G \colon \mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C}) \to \mathcal{C}$ such that $G \circ F$ is naturally isomorpic to the identity functor on $\mathcal{C}$ and $F \circ G$ is naturally isomorphic to the identity functor on $\mathbf{End}_{\mathrm{mod}-\mathcal{C}}(\mathcal{C})$. You do not need to endow G with the structure of a monoidal functor or check that the natural isomorphisms are monoidal. (But you are still encouraged to do this for yourself.)

Exercise 2. – *to be handed in by 3 March 2023*

Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories and let $(F, \varphi, \varepsilon) \colon \mathcal{C} \to \mathcal{D}$ and $(G, \varphi', \varepsilon') \colon \mathcal{C} \to \mathcal{D}$ be monoidal functors. Further let $u \colon F \to G$ be a monoidal natural transformation. Show that u is an isomorphism of functors.

You can follow the sequence of hints below.

We first make the following definition:

Definition. A *contragrediant* of a homomorphism $f \colon X \to Y$ in $\mathcal{C}$ is a homomorphism $f^\vee \colon X^* \to Y^*$ such that

$$\mathrm{ev}_Y \circ (f^\vee \otimes f) = \mathrm{ev}_X \qquad \text{and} \qquad (f \otimes f^\vee) \circ \mathrm{coev}_X = \mathrm{coev}_Y.$$

Now you can proceed as follows:

(a) Let $f \colon X \to Y$ be a homomorphism in $\mathcal{C}$ with a contragredient $f^\vee \colon X^* \to Y^*$. Show that

$$f^* \circ f^\vee = \mathrm{id}_{X^*} \qquad \text{and} \qquad f^\vee \circ f^* = \mathrm{id}_{Y^*}.$$

Hint: Recall that $\mathrm{id}_{X^*} = (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X)$ by the zig-zag relation and

$$f^* = (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X)$$

by definition.

(b) Show that $u_{X^*} \colon F(X)^* = F(X^*) \to G(X^*) = G(X)^*$ is a contragredient of $u_X \colon F(X) \to G(X)$, for all objects X of $\mathcal{C}$.

Hint: Use the definition of monoidal natural transformations and of the evaluation and coevaluation maps for $F(X)$; see Definition 1.14 and the proof of Lemma 4.8 in the lecture notes.

(c) Conclude that u_X is an isomorphism for every object X of $\mathcal{C}$.

Exercise 3. – *to be handed in by 14 April 2023*

Let G be a finite group and consider the category $\mathbf{Vect}_{\Bbbk}^G$ with the forgetful functor

$$F \colon \mathbf{Vect}_{\Bbbk}^G \longrightarrow \mathbf{Vect}_{\Bbbk}.$$

(a) Show that $\mathrm{End}(F)$ is isomorphic to the bialgebra $\mathrm{Fun}(G, \Bbbk)$ of $\Bbbk$-valued functions on G, where the counit $\varepsilon\colon \mathrm{Fun}(G, \Bbbk) \to \Bbbk$ and the comultiplication

$$\delta\colon \mathrm{Fun}(G, \Bbbk) \longrightarrow \mathrm{Fun}(G, \Bbbk) \otimes \mathrm{Fun}(G, \Bbbk) \cong \mathrm{Fun}(G \times G, \Bbbk)$$

are defined by

$$\varepsilon(f) = f(e) \qquad \text{and} \qquad \delta(f)(g, h) = f(gh)$$

respectively, for $f \in \mathrm{Fun}(G, \Bbbk)$ and $g, h \in G$.

Remark: In this exercise, you may assume without proof that the canonical homomorphism

$$\mathrm{End}(F) \otimes \mathrm{End}(F) \longrightarrow \mathrm{End}(F \times F)$$

is an isomorphism, and that the comultiplication on $\mathrm{End}(F)$ *arising by duality from the multiplication on* $\mathrm{End}^\vee(F)$ *sends a natural endomorphism* φ *of* F *to the natural endomorphism* $\tilde{\varphi}$ *of* $F \times F$ *with components*

$$\tilde{\varphi}_{X,Y}\colon F(X) \otimes F(Y) \cong F(X \otimes Y) \xrightarrow{\varphi_{X \otimes Y}} F(X \otimes Y) \cong F(X) \otimes F(Y).$$

Similarly, you may assume that the counit on $\mathrm{End}(F)$ *is given by*

$$\varphi \longmapsto \varphi_{\mathbf{1}} \in \mathrm{End}_{\Bbbk}\big(F(\mathbf{1})\big) \cong \Bbbk.$$

(b) Show that the dual bialgebra $\mathrm{Fun}(G, \Bbbk)^*$ of $\mathrm{Fun}(G, \Bbbk)$ is isomorphic (as a bialgebra) to the group algebra $\Bbbk[G]$.

(c) Conclude that $\mathbf{Vect}_{\Bbbk}^G$ is monoidally equivalent to the category of $\Bbbk[G]$-comodules.